Analysis of Boolean Functions	**Prof. Ryan O'Donnell**
Chapter 9: Basic of Hypercontractivity	
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Table 1: Problem Set 6: Progress

Problem 1. Exercise 9.3 (Tightness of Bonami Lemma)

Solution.

(a) Let $A := \{S_1 \times S_2 \times \cdots \times S_7 \in \binom{[n]}{\frac{k}{3}, \frac{k}{3}, \dots, \frac{k}{3}, n-2k}\}$, we have

$$\begin{aligned}
\mathbb{E}[f^4] &= \mathbb{E}_{x\in\{-1,1\}^n}[\sum_{T_1,\dots,T_4\subseteq[n],|T_1|=\cdots=|T_4|=k} x^{T_1}x^{T_2}x^{T_3}x^{T_4}] && (1)\\
&= \sum_{T_1,\dots,T_4\in[n],|T_1|=\cdots=|T_4|=k} \mathbb{E}_{x\in\{-1,1\}^n}[x^{T_1}x^{T_2}x^{T_3}x^{T_4}] && (2)\\
&\geq \sum_{S_1\times\cdots\times S_7\in A} \mathbb{E}_{x\in\{-1,1\}^n}[x^{S_1\cup S_2\cup S_3}x^{S_4\cup S_5\cup S_6}x^{S_1\cup S_3\cup S_5}x^{S_2\cup S_4\cup S_6}] && (3)\\
&= \sum_{S_1\times\cdots\times S_7\in A} \mathbb{E}_{x\in\{-1,1\}^n}[(x^{\cup_{i\in[6]}S_i})^2] && (4)\\
&= |A| = \binom{n}{\frac{k}{3}, \frac{k}{3}, \dots, \frac{k}{3}, n-2k}. && (5)
\end{aligned}$$

Also,

$$\begin{aligned}
\mathbb{E}[f^2] &= \mathbb{E}_{x\in\{-1,1\}^n}[\sum_{T_1,T_2\subseteq[n],|T_1|=|T_2|=k} x^{T_1}x^{T_2}] && (6)\\
&= \sum_{T_1,T_2\subseteq[n],|T_1|=|T_2|=k} \mathbb{E}_{x\in\{-1,1\}^n}[x^{T_1}x^{T_2}] && (7)\\
&= \sum_{T\subseteq[n],|T|=k} 1 = \binom{n}{k}. && (8)
\end{aligned}$$

As a result, we have

$$\frac{\mathbb{E}[f^4]}{\binom{n}{\frac{k}{3},\frac{k}{3},\dots,\frac{k}{3},n-2k}} \geq 1 = \frac{\mathbb{E}[f^2]^2}{\binom{n}{k}^2}. \quad (9)$$

(b) Expand the formula, we have

$$\frac{\binom{n}{\frac{k}{3},\dots,\frac{k}{3},n-2k}}{\binom{n}{k}^2} = \frac{(n)_{2k}\cdot(k!)^2}{(n)_k^2\cdot(\frac{k}{3}!)^6}, \quad (10)$$

where $(n)_k = n\cdot(n-1)\cdots(n-k+1)$. By Stirling's formula and taking $n\to\infty$, we have

$$\begin{aligned}
\frac{(k!)^2}{(\frac{k}{3}!)^6} &= \Theta(k^{-2}9^k), && (11)\\
\lim_{n\to\infty}\frac{(n)_{2k}}{(n)_k^2} &= 1. && (12)
\end{aligned}$$

Plug-in (a), we have $\|f\|_4 \geq \Omega(k^{-1/2})\cdot\sqrt{3}^k\|f\|_2$.

Problem 2. Exercise 9.6 (Bonami Lemma implies (2,4)-Hypercontractivity)

Solution.

(a) By triangle inequality, Bonami lemma, and the fact that $\|f^{=k}\|_2 \le \|f\|_2$ for any $k \in \mathbb{N}$, we have

$$\|T_{(1-\delta)/\sqrt{3}}f\|_4 \le \sum_{k=0}^{\infty} \|T_{(1-\delta)/\sqrt{3}}f^{=k}\|_4 \tag{13}$$

$$(\because \text{Bonami lemma}) \le \sum_{k=0}^{\infty} \sqrt{3}^k \|T_{(1-\delta)/\sqrt{3}}f^{=k}\|_2 \tag{14}$$

$$= \sum_{k=0}^{\infty}(1-\delta)^k \|f^{=k}\|_2 \tag{15}$$

$$\le \sum_{k=0}^{\infty}(1-\delta)^k \|f\|_2 \le \frac{1}{\delta}\|f\|_2. \tag{16}$$

Note that $\sum_{k=0}^{\infty} \|f^{=k}\|_2^2 = \|f\|_2^2$ by Parseval's equality.

(b) Here, we show that for any $h : \{-1,1\}^n \to \mathbb{R}$, $\|g^{\oplus d}\|_p = \|g\|_p^d$ for any $p \in \mathbb{R}^+$ and $d \in \mathbb{N}^+$.

$$\|g^{\oplus d}\|_p^p = \sum_{x^{(1)},x^{(2)},\ldots,x^{(d)} \in \{-1,1\}^n} g(x^{(1)})^p g(x^{(2)})^p \cdots g(x^{(d)})^p \tag{17}$$

$$= \prod_{i=1}^{d} \Big(\sum_{x^{(i)} \in \{-1,1\}^n} g(x^{(i)})^p \Big) \tag{18}$$

$$= \|g\|_p^{dp}. \tag{19}$$

(c) Apply (a) on $T_{(1-\delta)/\sqrt{3}}f^{\oplus d}$ we get

$$\|T_{(1-\delta)/\sqrt{3}}f\|_4^d = \|T_{(1-\delta)/\sqrt{3}}f^{\oplus d}\|_4 \tag{20}$$

$$\le \frac{1}{\delta}\|f^{\oplus d}\|_2 = \frac{1}{\delta}\|f\|_2^d. \tag{21}$$

That is, $\|T_{(1-\delta)/\sqrt{3}}f\|_4 \le (\frac{1}{\delta})^{1/d}\|f\|_2$. When $d \to \infty$, we have $\|T_{(1-\delta)/\sqrt{3}}f\|_4 \le \|f\|_2$.

(d) Take $\delta \to 0^+$, we have $\|T_{1/\sqrt{3}}f\|_4 \le \|f\|_2$, *i.e.,* the (2,4)-Hypercontractivity Theorem.

Problem 3. Exercise 9.9 (Basic properties of (p, q, ρ)-hypercontractivity)

Solution.

(a) For any $a, b \in \mathbb{R}$,

$$\begin{aligned} \|a + \rho b(c\mathbf{X})\|_q &= \|a + \rho bc\mathbf{X}\|_q && (22) \\ &\leq \|a + bc\mathbf{X}\|_p && (23) \\ &= \|a + b(c\mathbf{X})\|_p. && (24) \end{aligned}$$

Thus, $c\mathbf{X}$ is (p, q, ρ)-hypercontractive.

(b) Take $a = 0$ and $b = 1$, by the (p, q, ρ)-hypercontractivity of $\mathbf{X}$, we have

$$\begin{aligned} \rho\|\mathbf{X}\|_q &= \|\rho\mathbf{X}\|_q && (25) \\ &\leq \|\mathbf{X}\|_p. && (26) \end{aligned}$$

Thus, $\rho \leq \frac{\|\mathbf{X}\|_p}{\|\mathbf{X}\|_q}$.

Problem 4. Exercise 9.10 (Basic properties of (p, q, ρ)-hypercontractivity)

Solution.

(a) By Taylor expansion to the first order, we have

$$\|1 + b\mathbf{X}\|_r \le 1 + b\mathbb{E}[\mathbf{X}] + O(b^2). \tag{27}$$

As $\mathbf{X}$ is (p, q, ρ)-hypercontractive, we have $\|1 + \rho\epsilon\mathbf{X}\|_q \le \|1 + \epsilon\mathbf{X}\|_p$. Combine with (27), we have

$$1 + \rho\epsilon\mathbb{E}[\mathbf{X}] + O(\rho^2\epsilon^2) \le 1 + \epsilon\mathbb{E}[\mathbf{X}] + O(\epsilon^2). \tag{28}$$

By letting $\epsilon \to 0^+$, as $0 < \rho < 1$, there exists ϵ_0 such that

$$\rho\epsilon_0\mathbb{E}[\mathbf{X}] \le \epsilon_0\mathbb{E}[\mathbf{X}]. \tag{29}$$

As $\rho <$, the above inequality holds only if $\mathbb{E}[\mathbf{X}] = 0$.

(b) To Taylor expand $\|1 + b\mathbf{X}\|_r$, let's compute $\frac{d^2}{d(b\mathbf{X}_i)^2}\|1 + b\mathbf{X}\|_r$ as follows.

$$\frac{d}{d\mathbf{X}_i}\|1 + b\mathbf{X}\|_r = b\|1 + b\mathbf{X}\|_r^{1-r}(1 + b\mathbf{X}_i)^{r-1}, \tag{30}$$

$$\frac{d^2}{d\mathbf{X}_i^2}\|1 + b\mathbf{X}\|_r = (1 - r)b^2 \cdot \|1 + b\mathbf{X}\|_r^{1-2r} \cdot (1 + b\mathbf{X}_i)^{2r-1} + b(r - 1)\|1 + b^2\mathbf{X}\|_r^{1-r} \cdot (1 + b\mathbf{X}_i)^{r-2}. \tag{31}$$

Thus, Taylor expand $\|1 + b\mathbf{X}\|_r$ to the second order, we have

$$\|1 + b\mathbf{X}\|_r = 1 + b\mathbb{E}[\mathbf{X}] + \frac{b^2(r-1)}{2}\mathbb{E}[\mathbf{X}^2] + O(b^3) \tag{32}$$

From (a), we know that $\mathbb{E}[\mathbf{X}] = 0$ so by the (p, q, ρ)-hypercontractivity of $\mathbf{X}$, we have

$$1 + \frac{\rho^2\epsilon^2(q-1)}{2}\mathbb{E}[\mathbf{X}^2] + O(\epsilon^3\rho^3) \le 1 + \frac{\epsilon^2(p-1)}{2}\mathbb{E}[\mathbf{X}^2] + O(\epsilon^3). \tag{33}$$

Similarly, when $\epsilon \to 0^+$, there exists $\epsilon_0 > 0$ such that

$$\rho^2\epsilon_0^2(q-1)\mathbb{E}[\mathbf{X}^2] \le \epsilon_0^2(p-1)\mathbb{E}[\mathbf{X}^2]. \tag{34}$$

That is, $\rho \le \sqrt{\frac{p-1}{q-1}}$.

Problem 5. Exercise 9.11 (Basic properties of (p, q, ρ)-hypercontractivity)

Solution.

(a) For any $a, b \in \mathbb{R}$,

$$\begin{aligned}\|a + 0 \cdot b\mathbf{X}\|_q &= a &(35)\\ &= \mathbb{E}[a + b\mathbf{X}] &(36)\\ &\leq \mathbb{E}[|a + b\mathbf{X}|] &(37)\\ &= \|a + b\mathbf{X}\|_1 &(38)\\ &\leq \|a + b\mathbf{X}\|_q. &(39)\end{aligned}$$

Thus, $\mathbf{X}$ is $(q, q, 0)$-hypercontractive.

(b) WLOG, consider arbitrary $a \in \mathbb{R}$,

$$\begin{aligned}\|a + \rho\mathbf{X}\|_q &= \|(1-\rho)a + \rho(a + \mathbf{X})\|_q &(40)\\ &\leq (1-\rho)a + \rho\|a + \mathbf{X}\|_q &(41)\\ &\leq (1-\rho)\|a + \mathbf{X}\|_q + \rho\|a + \mathbf{X}\|_q &(42)\\ &= \|a + \mathbf{X}\|_q. &(43)\end{aligned}$$

Thus, $\mathbf{X}$ is (q, q, ρ)-hypercontractive for any $0 \leq \rho \leq 1$.

(c) By Exercise 9.10 (a), $\mathbb{E}[\mathbf{X}] = 0$. Let $0 \leq \rho'\rho \leq 1$. By Exercise 9.11 (b), $\mathbf{X}$ is $(q, q, \rho'/\rho)$-hypercontractive. That is, for any $a, b \in \mathbb{R}$,

$$\begin{aligned}\|a + \rho' b\mathbf{X}\|_q &\leq \|a + \rho b\mathbf{X}\|_q &(44)\\ &\leq \|a + b\mathbf{X}\|_p, &(45)\end{aligned}$$

where the first inequality is due to the $(q, q, \rho'/\rho)$-hypercontractivity and the second inequality is due to the (p, q, ρ)-hypercontractive. Thus, if $\mathbf{X}$ is (p, q, ρ)-hypercontractive, then $\mathbf{X}$ is also (p, q, ρ')-hypercontractive for any $0 \leq \rho' \leq \rho$.

Problem 6. Exercise 9.14 ($(p,2)$-Bonami Lemma)

Solution.

(a) For any $1 \le p \le 2$, take $q = \frac{p}{p-1} \ge 2$. By Theorem 9.21 (the $(2,q)$-Bonami Lemma), we have

$$\|f\|_q \le \sqrt{q-1}^k \|f\|_2. \tag{46}$$

By Holder's inequality, we have

$$\|f\|_2^2 \le \|f\|_p \cdot \|f\|_q. \tag{47}$$

Combine the above two equations, we have the $(p,2)$-Bonami Lemma as follows.

$$\|f\|_2 \le \sqrt{\frac{1}{p-1}}^k \|f\|_p. \tag{48}$$

(b) It suffices to show that $\exp(2-\frac{4}{p}) > p-1$ for any $1 \le p < 2$. First, observe that when $p=2$, the two sides are both equal to 1. Next, differentiate both sides and get

$$\frac{d}{dp}\text{LHS} = \frac{4}{p^2}\exp(2-\frac{4}{p}), \tag{49}$$

$$\frac{d}{dp}\text{RHS} = 1. \tag{50}$$

By first order Taylor approximation,

$$e^{\frac{2}{p}-1} \ge 1 + \frac{2}{p} - 1 = \frac{2}{p}. \tag{51}$$

Plug-in (49) and (50), we have $\frac{d}{p}\text{LHS} \le \frac{d}{dp}\text{RHS}$ and thus the inequality holds.

Problem 7. Exercise 9.15 (Generality of Theorem 9.22)

Solution. Similarly to the proof of Theorem 9.22, take $1 \leq p \leq 2$ such that $\frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{2+\epsilon}$ for some θ and $\epsilon > 0$. By the interpolation version of Holder's inequality, we have

$$\|f\|_2 \leq \|f\|_p^\theta \cdot \|f\|_{2+\epsilon}^{1-\theta} \tag{52}$$

$$\leq \|f\|_p^\theta \cdot \sqrt{1+\epsilon}^{k(1-\theta)} \|f\|_2^{1-\theta}. \tag{53}$$

That is,

$$\|f\|_2 \leq (1+\epsilon)^{\frac{1-\theta}{2\theta}k} \|f\|_p. \tag{54}$$

Compute that

$$\theta = \frac{\epsilon p}{2(2+\epsilon-p)}, \tag{55}$$

$$\frac{1-\theta}{2\theta} = (\frac{2}{p}-1)\frac{1}{\epsilon} + \frac{1}{p} - \frac{1}{2}. \tag{56}$$

Let $\epsilon \to 0^+$, we have

$$\|f\|_2 \leq e^{k(\frac{2}{p}-1)} \|f\|_p. \tag{57}$$

Problem 8. Exercise 9.16 ($(2,q)$-Hypercontractivity by induction of derivative)

Solution. Let's prove the $(2,q)$-Hypercontractivity theorem using induction of derivative. Recall that $f = \mathbf{e} + \mathbf{x}_n \mathbf{d}$. That is, by the linearity of T, we can rewrite $\|Tf\|_q$ as follows.

$$\begin{aligned} \|Tf\|_q^2 &= \|T\mathbf{e} + (1/\sqrt{q-1})\mathbf{x}_n T\mathbf{d}\|_q^2 & (58)\\ &= \mathbb{E}_{\mathbf{x}'}[\mathbb{E}_{\mathbf{x}_n}[|T\mathbf{e} + (1/\sqrt{q-1})\mathbf{x}_n T\mathbf{d}|^q]]^{2/q} & (59)\\ &= \mathbb{E}_{\mathbf{x}'}[\mathbb{E}_{\mathbf{x}_n}[\Big(|T\mathbf{e} + (1/\sqrt{q-1})\mathbf{x}_n T\mathbf{d}|^2\Big)^{q/2}]]^{2/q} & (60)\end{aligned}$$

As $(1/\sqrt{q-1}) \leq 1$ and $\mathbf{x}_n \in \{-1, 1\}$, we have

$$\mathbb{E}_{\mathbf{x}_n}[|T\mathbf{e} + (1/\sqrt{q-1})\mathbf{x}_n T\mathbf{d}|^2] = (T\mathbf{e})^2 + \frac{1}{q-1}(T\mathbf{d})^2 \leq (T\mathbf{e})^2 + (T\mathbf{d})^2. \tag{61}$$

Since $h(t) = t^{q/2}$ is convex when $q \geq 2$, we have

$$\mathbb{E}_{\mathbf{x}_n}[\Big(|T\mathbf{e} + (1/\sqrt{q-1})\mathbf{x}_n T\mathbf{d}|^2\Big)^{q/2}] \leq \Big(\mathbb{E}_{\mathbf{x}_n}[|T\mathbf{e} + (1/\sqrt{q-1}\mathbf{x}_n T\mathbf{d})|^2]\Big)^{q/2}. \tag{62}$$

Combine (61) and (62), we have

$$\mathbb{E}_{\mathbf{x}'}[\mathbb{E}_{\mathbf{x}_n}[|T\mathbf{e} + (1/\sqrt{q-1})\mathbf{x}_n T\mathbf{d}|^q]]^{2/q} \leq \mathbb{E}_{\mathbf{x}'}[\Big((T\mathbf{e})^2 + (T\mathbf{d})^2\Big)^{q/2}]^{2/q} = \|(T\mathbf{e})^2 + (T\mathbf{d})^2\|_{q/2}. \tag{63}$$

By the triangle inequality of $\|\cdot\|_{q/2}$, we have

$$\|(T\mathbf{e})^2 + (T\mathbf{d})^2\|_{q/2} \leq \|(T\mathbf{e})^2\|_{q/2} + \|(T\mathbf{d})^2\|_{q/2} = \|T\mathbf{e}\|_q^2 + \|T\mathbf{d}\|_q^2. \tag{64}$$

By induction hypothesis, $\|T\mathbf{e}\|_{q/2}^2 \leq \|\mathbf{e}\|_q^2$ and $\|T\mathbf{d}\|_2^2 \leq \|\mathbf{d}\|_2^2$. Thus, we conclude that

$$\|Tf\|_q^2 \leq \|\mathbf{e}\|_2^2 + \|\mathbf{d}\|_2^2 = \|f\|_2^2. \tag{65}$$

Problem 9. Exercise 9.18 (Sharpen Level-1 Inequality)

Solution.

(a) Combine the fact that $\mathbb{E}[f] = \alpha$ and the expansion of stability of f, for any $0 < \rho \leq 1$, we have

$$\mathbf{Stab}_\rho[f] = \sum_{k=0}^{n} \rho^k \mathbf{W}^k[f] \tag{66}$$
$$\geq \rho \mathbf{W}^1[f] + \mathbb{E}[f]^2 = \rho \mathbf{W}^1[f] + \alpha^2. \tag{67}$$

Also, by the Small-Set Expansion theorem,

$$\mathbf{Stab}_\rho[f] \leq \alpha^{\frac{2}{1-\rho}}. \tag{68}$$

Thus, $\mathbf{W}^1[f] \leq \frac{1}{\rho}(\alpha^{\frac{2}{1-\rho}} - \alpha^2)$.

(b) Reformulate the RHS in (a), we have

$$\frac{1}{\rho}(\alpha^{\frac{2}{1-\rho}} - \alpha^2) = \alpha^2 \frac{1}{\rho}(\alpha^{-\frac{2\rho}{1-\rho}} - 1) \tag{69}$$
$$= \alpha^2 \frac{1}{\rho}(1 - \frac{2\rho \ln \alpha}{1-\rho} + O(\rho^2) - 1) \tag{70}$$
$$= \alpha^2 (\frac{2\ln(1/\alpha)}{1-\rho} + O(\rho)). \tag{71}$$

Let $\rho \to 0$ and combine with (a), we have $\mathbf{W}^1[f] \leq 2\alpha^2 \ln(1/\alpha)$.

Problem 10. Exercise 9.19 (Level-k inequalities)

Solution.

(a) Plug-in $k_\epsilon = \frac{2e}{1-\epsilon}\ln(1/\alpha)$, we have

$$\begin{aligned}
\mathbf{W}^{\leq k}[f] &\leq \left(\frac{2e}{2\ln(1/\alpha)(1-\epsilon)}\ln(1/\alpha)\right)^{2(1-\epsilon)\ln(1/\alpha)}\alpha^2 && (72)\\
&= \left(\frac{e}{1-\epsilon}\right)^{2(1-\epsilon)\ln(1/\alpha)}\alpha^2 && (73)\\
&= \alpha^{2-2(1-\epsilon)(1-\ln(1-\epsilon))} && (74)\\
&= \alpha^{2\epsilon-2(1-\epsilon)\ln(1/(1-\epsilon))}. && (75)
\end{aligned}$$

(b) By second order Taylor expansion, we have

$$\ln(1-\epsilon) \leq -\epsilon + \frac{\epsilon^2}{2}. \qquad (76)$$

Thus,

$$\begin{aligned}
\alpha^{2\epsilon-2(1-\epsilon)\ln(1/(1-\epsilon))} &= \alpha^{2\epsilon+2(1-\epsilon)\ln((1-\epsilon))} && (77)\\
&\leq \alpha^{2\epsilon+2(1-\epsilon)(-\epsilon+\epsilon^2/2)} && (78)\\
&= \alpha^{2\epsilon^2-\epsilon^2} = \alpha^{\epsilon^2}. && (79)
\end{aligned}$$

Problem 11. Exercise 9.20 (KKL fails for function with range $[-1,1]$)

Solution. Consider $f(x) = \text{trunc}(\frac{\sum_{i\in[n]} x_i}{\sqrt{n}})$ and let $f^*(x) = \frac{\sum_{i\in[n]} x_i}{\sqrt{n}}$. Observe that

$$\mathrm{V}ar[f] \geq \mathbb{P}[|\frac{\sum_{i\in[n]} \mathbf{x}_i}{\sqrt{n}}| > 1] = \Omega(1). \tag{80}$$

Also, for any $i \in [n]$,

$$\mathbf{Inf}_i[f] \leq \mathbf{Inf}_i[f^*]. \tag{81}$$

As $\mathbf{Inf}_i[f^*] = \mathbb{E}[\mathbf{x}_i][(D_i f^*)^2] = \frac{1}{n}$, we have

$$\mathbf{MaxInf}[f] \leq \mathbf{MaxInf}[f^*] = \frac{1}{n}. \tag{82}$$

As a result, the statement of KKL doesn't hold on f.

Problem 12. Exercise 9.24 (Hamming ball achieves Small-Set Expansion stability bound) By Small-Set Expansion theorem, we have an upper bound for the ρ-stability of indicator function. Concretely, for any $A \subseteq \{-1,1\}^n$, $\mathbb{E}[\mathbf{1}_A] = \alpha$, we have

$$\mathbf{Stab}_\rho[\mathbf{1}_A] \leq \alpha^{\frac{2}{1+\rho}}. \tag{83}$$

In this exercise, we are going to show that the Hamming ball indicator function will asymptotically achieve the bound by proving that

$$\Lambda_\rho(\Phi^{-1}(t)) = \tilde{\Theta}(\mu^{\frac{2}{1+\rho}}). \tag{84}$$

Solution.

(a)

Problem 13. Exercise 9.25

Solution.

(a)

Problem 14. Exercise 9.29 (General-variance KKL)

Solution. It suffices to prove that $\sum_{S:|S|\geq 1}(\frac{1}{3})^{|S|}\widehat{f}(S)^2 \geq \mathrm{V}ar[f]\cdot 3^{-\mathbf{I}[f]/\mathrm{V}ar[f]}$. Recall that $\mathrm{V}ar[f] = \sum_{S:|S|\geq 1}\widehat{f}(S)^2$. Thus, by the concavity of $h(t) = (1/3)^t$, we have

$$\sum_{S:|S|\geq 1}\frac{\widehat{f}(S)^2}{\mathrm{V}ar[f]}(\frac{1}{3})^{|S|} \geq (\frac{1}{3})^{\sum_{S:|S|\geq 1}\frac{\widehat{f}(S)^2}{\mathrm{V}ar[f]}|S|} \tag{85}$$

$$= 3^{-\mathbf{I}[f]/\mathrm{V}ar[f]}. \tag{86}$$

Note that replace $\frac{1}{3}$ with $0 < \rho < 1$, we have

$$\sum_{S:|S|\geq 1}\rho^{|S|}\widehat{f}(S)^2 \geq \mathrm{V}ar[f]\cdot\rho^{\mathbf{I}[f]/\mathrm{V}ar[f]}. \tag{87}$$

Problem 15. Exercise 9.30 (State-of-the-art constant of KKL)

Solution.

(a) From Exercise 9.29, we have

$$\sum_{S:|S|\geq 1} \rho^{|S|}\widehat{f}(S)^2 \geq \mathrm{V}ar[f] \cdot \rho^{\tilde{\mathbf{I}}[f]}. \tag{88}$$

Note that the LHS is upper bounded by $\frac{1}{\rho}\mathbf{I}^{(\rho)}[f]$ because

$$\mathbf{I}^{(\rho)}[f] = \sum_{S} |S|\rho^{|S|-1}\widehat{f}(S)^2 \tag{89}$$

$$\geq \frac{1}{\rho} \cdot \sum_{S:|S|\geq 1} \rho^{|S|}\widehat{f}(S)^2. \tag{90}$$

Also, for each $i \in [n]$, by Corollary 9,25, we have

$$\mathbf{I}_i^{(\rho)}[f] \leq \mathbf{I}_i[f]^{\frac{2}{1+\rho}}. \tag{91}$$

Thus, we have

$$\mathbf{I}^{(\rho)}[f] = \sum_{i\in[n]} \mathbf{I}_i^{(\rho)}[f] \leq \sum_{i\in[n]} \mathbf{I}_i[f]^{\frac{2}{1+\rho}} \leq \mathbf{MaxInf}[f]^{\frac{1-\rho}{1+\rho}} \cdot \mathbf{I}[f]. \tag{92}$$

As a result, from (88), (90), and (92), we have

$$\mathbf{MaxInf}[f]^{\frac{1-\rho}{1+\rho}} \geq \frac{1}{\rho} \cdot \tilde{\mathbf{I}}[f]^{-1} \cdot \rho^{\tilde{\mathbf{I}}[f]}. \tag{93}$$

For any $0 < \delta < 1$, take $\rho = \frac{1-\delta}{1+\delta} \in (0,1)$ such that $\delta = \frac{1-\rho}{1+\rho}$. We have

$$\mathbf{MaxInf}[f] \geq (\frac{1+\delta}{1-\delta})^{\frac{1}{\delta}} (\frac{1}{\tilde{\mathbf{I}}[f]})^{\frac{1}{\delta}} \cdot (\frac{1-\delta}{1+\delta})^{\frac{1}{\delta}\cdot\tilde{\mathbf{I}}[f]}. \tag{94}$$

Now, consider how small the base of $-\tilde{\mathbf{I}}[f]$ can be.

$$(\frac{1+\delta}{1-\delta})^{1/\delta} > (1+\frac{2}{1/\delta})^{1/\delta}. \tag{95}$$

Note that the RHS is a non-decreasing function. Thus, it suffices to lower bound the case when $\delta \to 0$.

$$\lim_{\delta\to 0}(1+\frac{2}{1/\delta})^{1/\delta} = e^2. \tag{96}$$

As a result, we have for any $C > e^2$

$$\mathbf{MaxInf}[f] \geq \tilde{\Omega}(C^{-\tilde{\mathbf{I}}[f]}). \tag{97}$$

(b) Let's lower bound $(\frac{1-\delta}{1+\delta})^{1/\delta}$ for $0 < \delta < 1/2$. By the definition of e, we have

$$(\frac{1-\delta}{1+\delta})^{1/\delta} = \left((1 - \frac{2\delta}{1+\delta})^{1+\delta/\delta}\right)^{1/(1+\delta)} \quad (98)$$

$$\geq \left(e^{-2}\right)^{1/(1+\delta)}. \quad (99)$$

When $0 < \delta < 1/2$, by second order Taylor expansion, we have

$$\frac{1}{1+\delta} = (1+\delta)^{-1} \leq 1 - \delta + \frac{\delta^2}{2} \leq 1 + \frac{\delta^2}{2}. \quad (100)$$

Combine (99) and (100), we have

$$(\frac{1-\delta}{1+\delta})^{1/\delta} \leq e^{-2-\delta^2}. \quad (101)$$

Now, take $\delta = \frac{1}{2\tilde{\mathbf{I}}[f]^{1/3}}$, by (94), (96), and (101), we have

$$\mathbf{MaxInf}[f] \geq (\frac{1+\delta}{1-\delta})^{\frac{1}{\delta}} (\frac{1}{\tilde{\mathbf{I}}[f]})^{\frac{1}{\delta}} \cdot (\frac{1-\delta}{1+\delta})^{\frac{1}{\delta} \cdot \tilde{\mathbf{I}}[f]} \quad (102)$$

$$\geq e^2 \cdot (\frac{1}{\tilde{\mathbf{I}}[f]})^{2\tilde{\mathbf{I}}[f]^{1/3}} \cdot \exp(-2\tilde{\mathbf{I}}[f]) \cdot \exp(-\frac{\tilde{\mathbf{I}}[f]^{1/3}}{4}). \quad (103)$$

Problem 16. Exercise 9.31 (Delicate version of Theorem 9.28)

Solution. Similarly, we take the parameters in Theorem 9.28 in the following setting. For any $\eta > 0$, $0 \le \epsilon < 1$, and $k \ge 0$, define

$$\tau = \frac{\epsilon^{1+\eta}}{\mathbf{I}[f]^{1+\eta}} C(\eta)^{-k}, \qquad J = \{j \in [n] : \mathbf{I}_j[f] \ge \tau\}, \qquad \text{so } |J| \le \frac{\mathbf{I}[f]^{2+\eta}}{\epsilon^{1+\eta}} C(\eta)^k. \tag{104}$$

Also, define

$$\mathcal{F} = \{S : S \subseteq J\} \cup \{S : |S| > k\}. \tag{105}$$

Now, consider $0 < \rho < 1$ chosen later according to η. The idea is to upper bound $\sum_{i \notin J} \mathbf{I}_i^{(\rho)}[f]$ with the error and lower bound it with the Fourier mass not in $\mathcal{F}$. By Corollary 9.25

$$\sum_{i \notin J} \mathbf{I}_i^{(\rho)}[f] \le \sum_{i \notin J} \mathbf{I}[f]^{\frac{2}{1+\rho}} \tag{106}$$

$$\le \max_{i \notin J} \mathbf{I}_i[f]^{\frac{1-\rho}{1+\rho}} \cdot \sum_{i \notin J} \mathbf{I}_i[f] \tag{107}$$

$$\le \tau^{\frac{1-\rho}{1+\rho}} \cdot \mathbf{I}[f] = \frac{\epsilon^{(1+\eta)\cdot\frac{1-\rho}{1+\rho}}}{\mathbf{I}[f]^{(1+\eta)\cdot\frac{1-\rho}{1+\rho}-1}} C(\eta)^{-k\frac{1-\rho}{1+\rho}}. \tag{108}$$

As to the lower bound, we have

$$\sum_{i \notin J} \mathbf{I}_i^{(\rho)}[f] = \sum_{i \notin J} \sum_{S:\ i \in S} \rho^{|S|-1} \widehat{f}(S)^2 \tag{109}$$

$$= \sum_{S} |S \cap \bar{J}| \rho^{|S|-1} \widehat{f}(S)^2 \tag{110}$$

$$\ge \sum_{S \notin \mathcal{F}} \rho^{|S|-1} \widehat{f}(S)^2 \tag{111}$$

$$\ge \rho^{-k} \sum_{S \notin \mathcal{F}} \widehat{f}(S)^2. \tag{112}$$

Now, take $\rho = \frac{\eta}{2+\eta} = C(\eta)^{1/2}$, we have $1 + \eta = \frac{1+\rho}{1-\rho}$. As a result, (108) and (112) become

$$C(\eta)^{-k/2} \cdot \sum_{S \notin \mathcal{F}} \widehat{f}(S)^2 \le \sum_{i \notin J} \mathbf{I}_i^{(\rho)}[f] \le C(\eta)^{-k/(1+\eta)} \epsilon \tag{113}$$

That is,

$$\sum_{S \notin \mathcal{F}} \widehat{f}(S)^2 \le C(\eta)^{-k(\frac{1}{2} - \frac{1}{1+\eta})} \epsilon. \tag{114}$$

Problem 17.

Solution.

(a)

Problem 18.

Solution.

(a)

Problem 19.

Solution.

(a)

Problem 20.

Solution.

(a)