Tracking the ℓ_2 Norm with Constant Update Time

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Harvard University

APPROX 2019

Streaming Algorithms

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$\boxed{1}$

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1 **2**

$$f^{(2)} = \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array}$$

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$$f^{(3)} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

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1 2 4 2 5

$$f^{(5)} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

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$$f^{(6)} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

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1	**2**	**4**	**2**	**5**	**2**	**1**	**1**

$$f^{(8)} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

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$$f^{(9)} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

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Deterministic algorithm: $\Theta(\min\{m, n\})$ space.

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Faster time!?

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- (ϵ, δ)**-One-shot estimation**: Output σ_m s.t.

$$\Pr\left[\left|\sigma_m - \|f^{(m)}\|_2^2\right| > \epsilon\|f^{(m)}\|_2^2\right] \leq \delta\,.$$

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Linear Sketch

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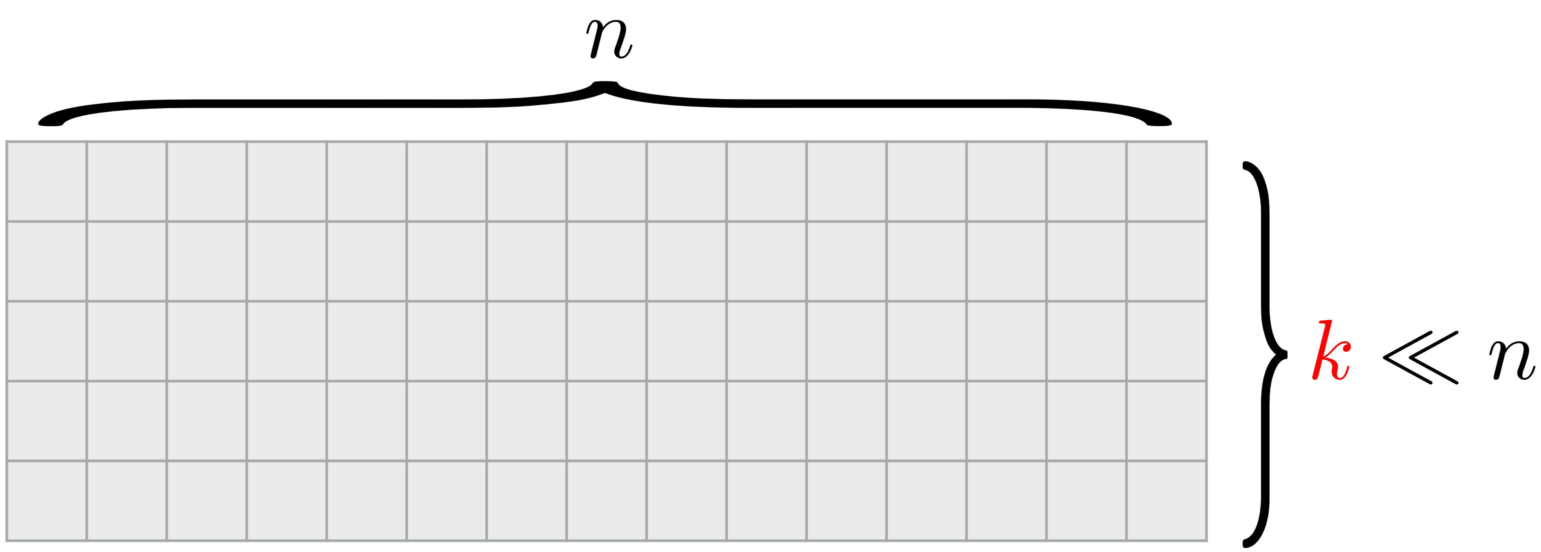

Sketching matrix Π

Linear Sketch

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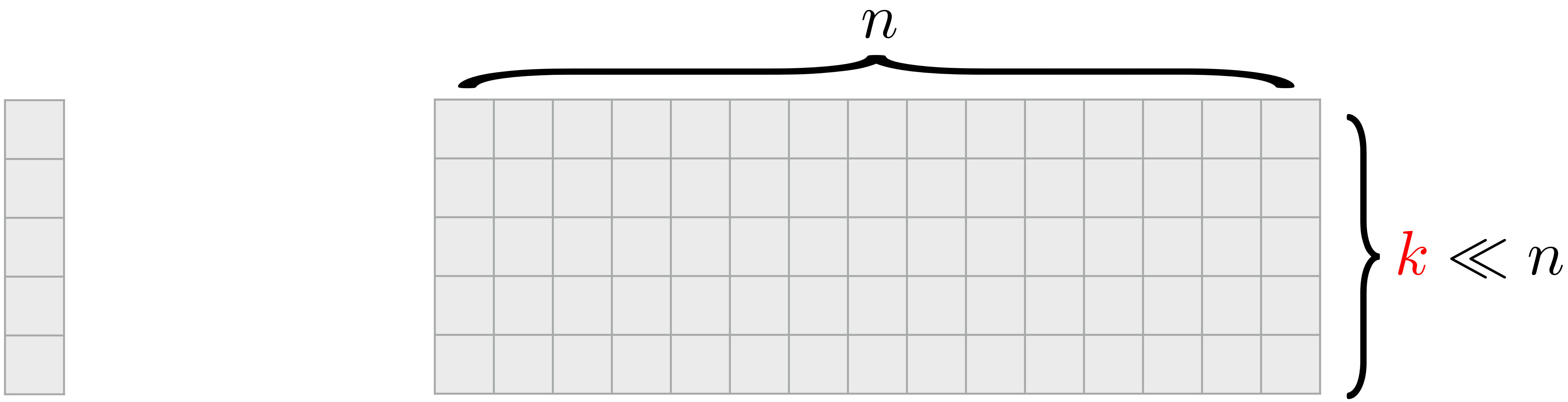

Sketching vector $\Pi f^{(t)}$

Sketching matrix Π

Linear Sketch

AMS Sketch
[Alon-Matias-Szegedy 96]

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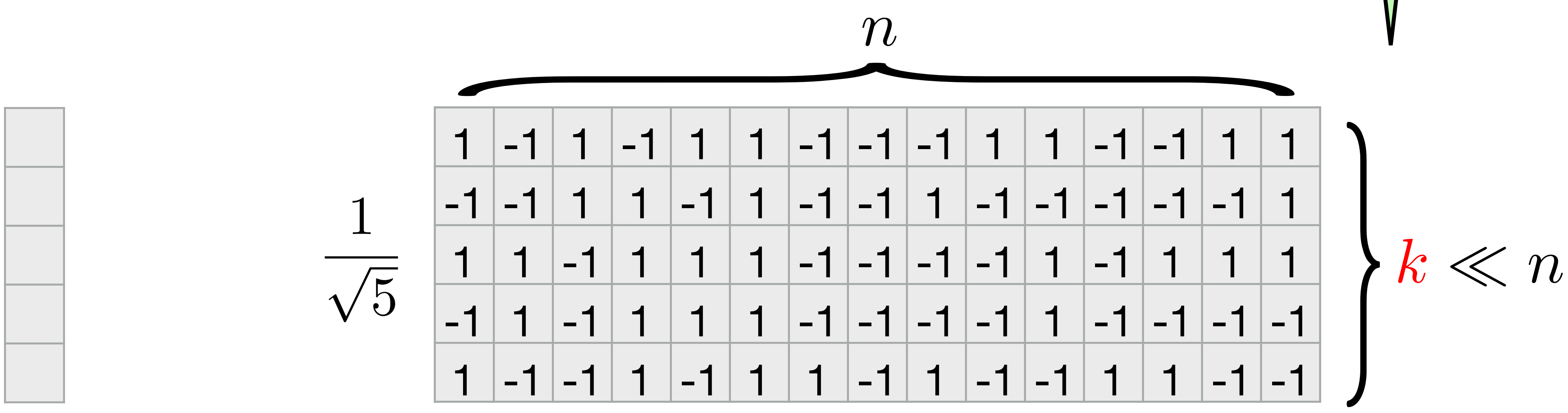

$$\frac{1}{\sqrt{5}} \left.\overbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ \hline -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ \hline 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ \hline -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ \hline 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ \hline \end{array}}^{n}\right\} k \ll n$$

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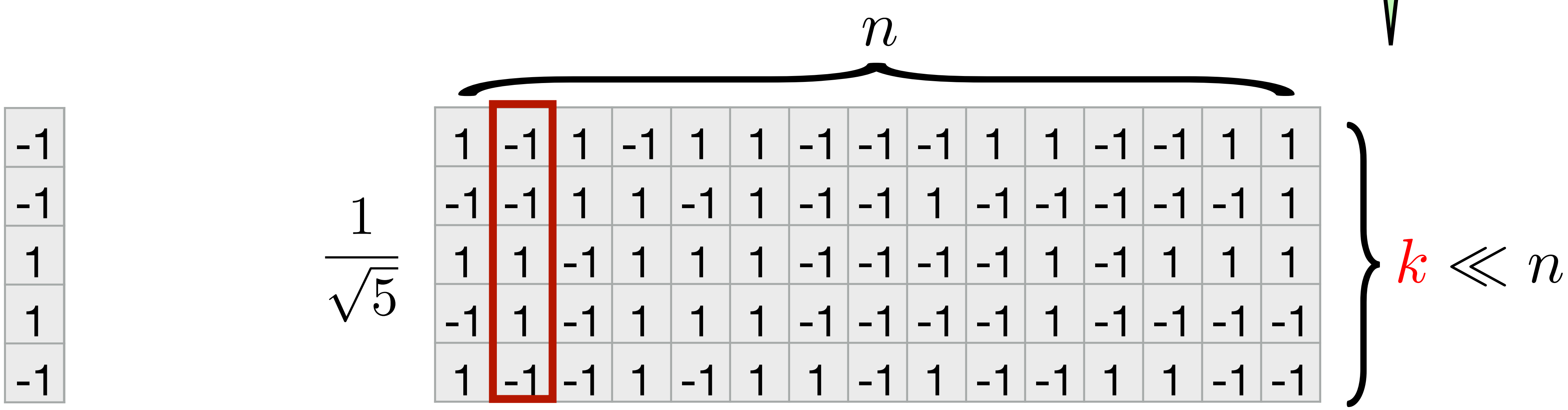

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \qquad \frac{1}{\sqrt{5}} \overbrace{\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix}}^{n} \Bigg\} \, k \ll n$$

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2

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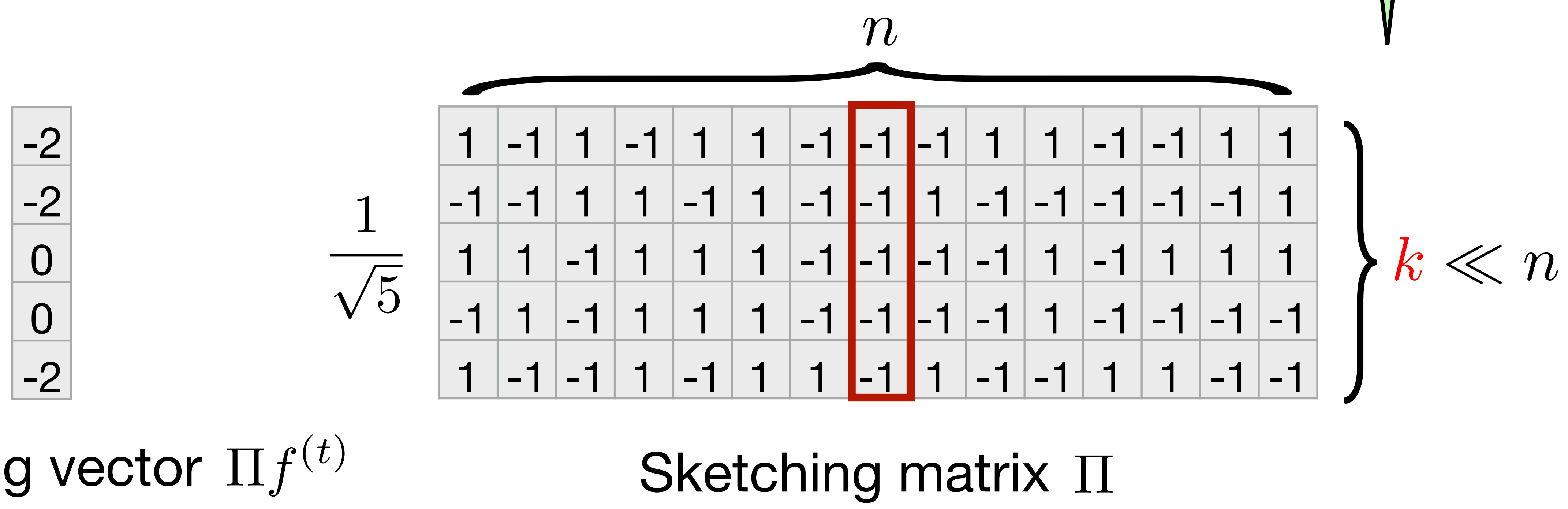

-2
-2
0
0
-2

Sketching vector $\Pi f^{(t)}$

$$\frac{1}{\sqrt{5}}$$

1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	-1	1	1
-1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	-1	1
1	1	-1	1	1	1	-1	-1	-1	-1	1	-1	1	1	1
-1	1	-1	1	1	1	-1	-1	-1	-1	1	-1	-1	-1	-1
1	-1	-1	1	-1	1	1	-1	1	-1	-1	1	1	-1	-1

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2 **8**

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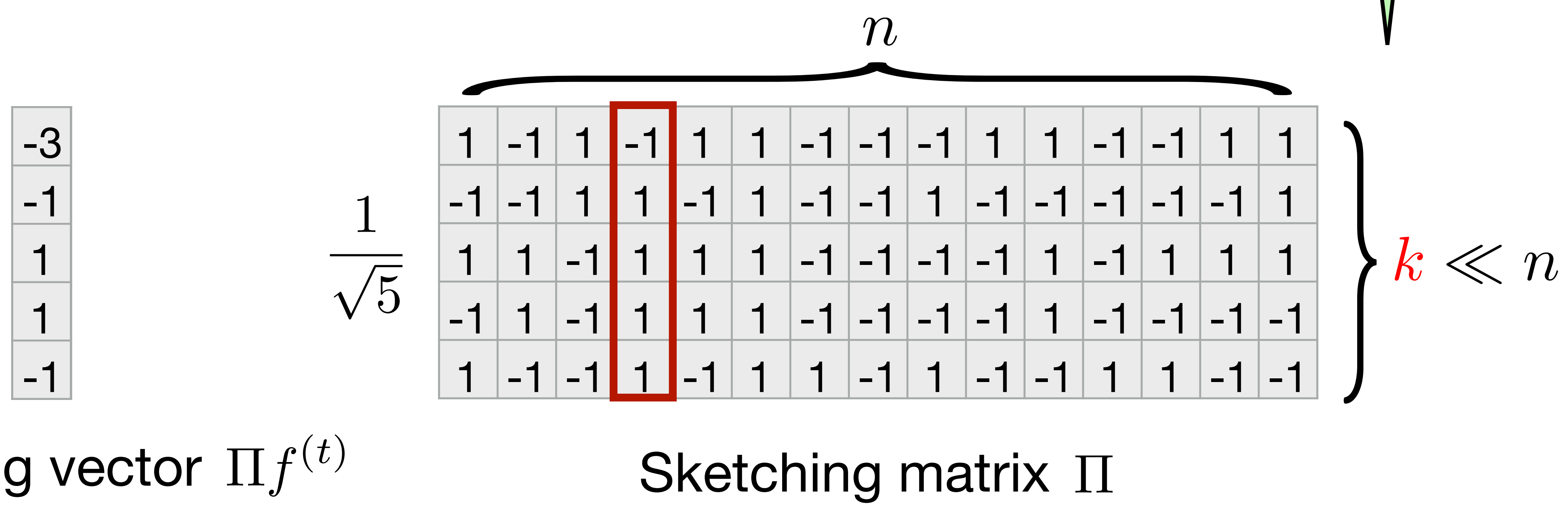

$$\begin{pmatrix} -3 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \qquad \frac{1}{\sqrt{5}} \overbrace{\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{pmatrix}}^{n} \left.\vphantom{\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}}\right\} k \ll n$$

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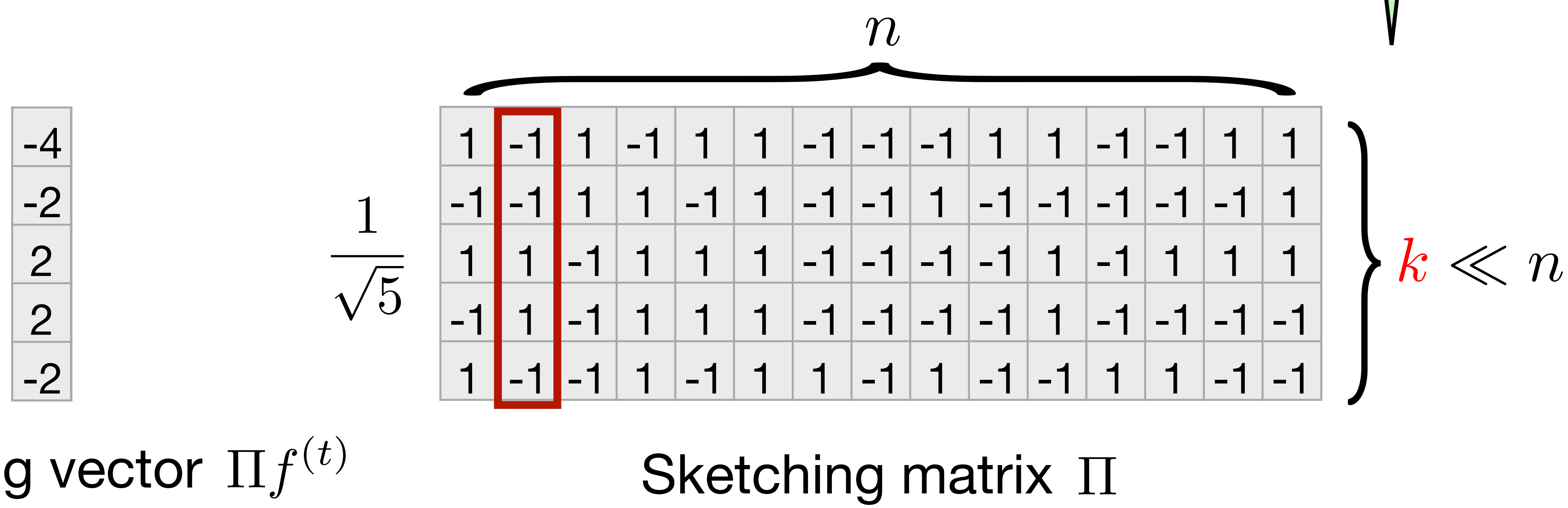

$$\begin{bmatrix}-4\\-2\\2\\2\\-2\end{bmatrix} \qquad \frac{1}{\sqrt{5}} \overbrace{\left[\begin{array}{c|c|ccccccccccccc} 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{array}\right]}^{n} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\end{array}}\right\} {\color{red}k} \ll n$$

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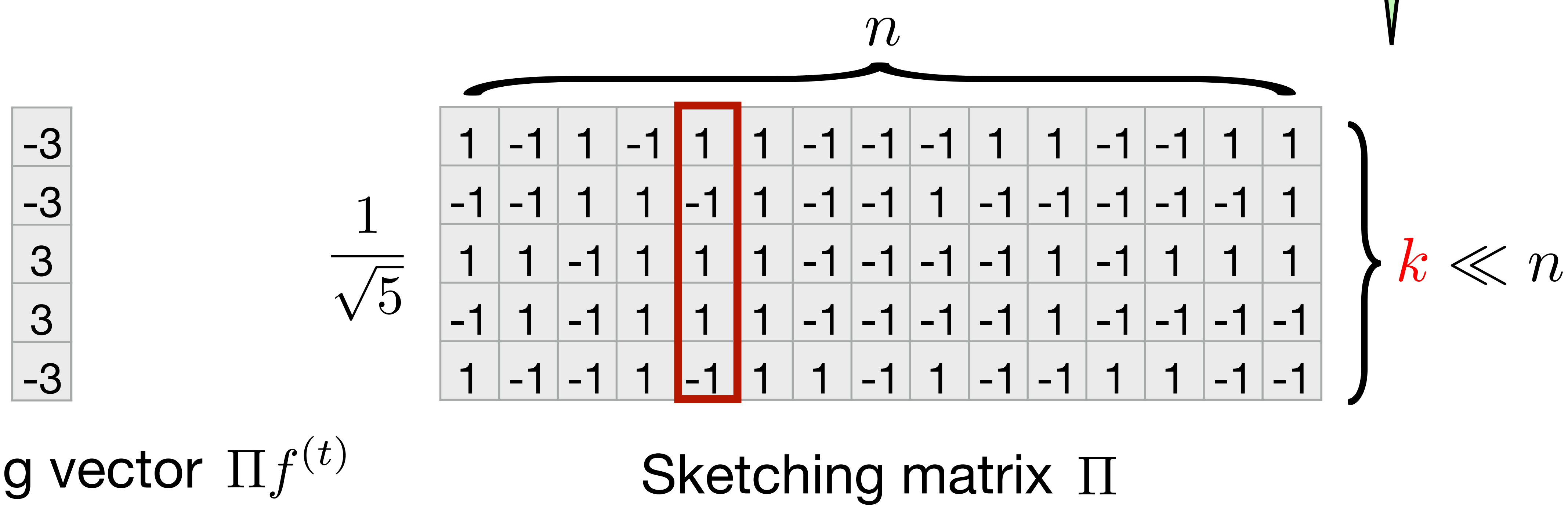

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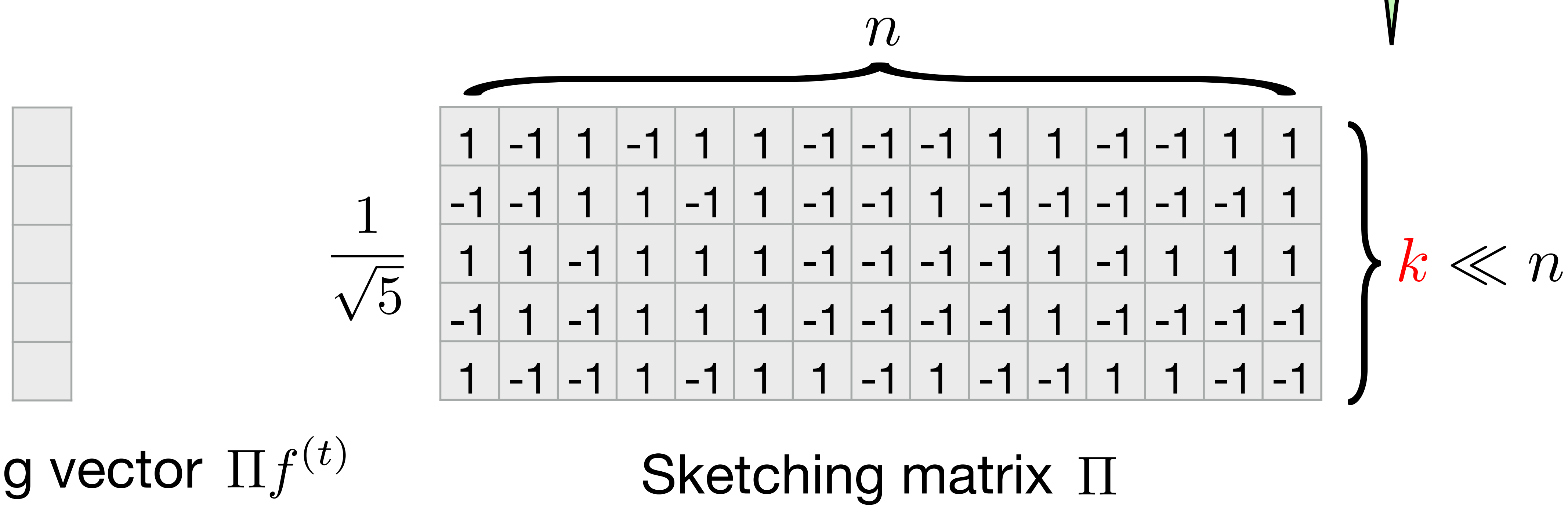

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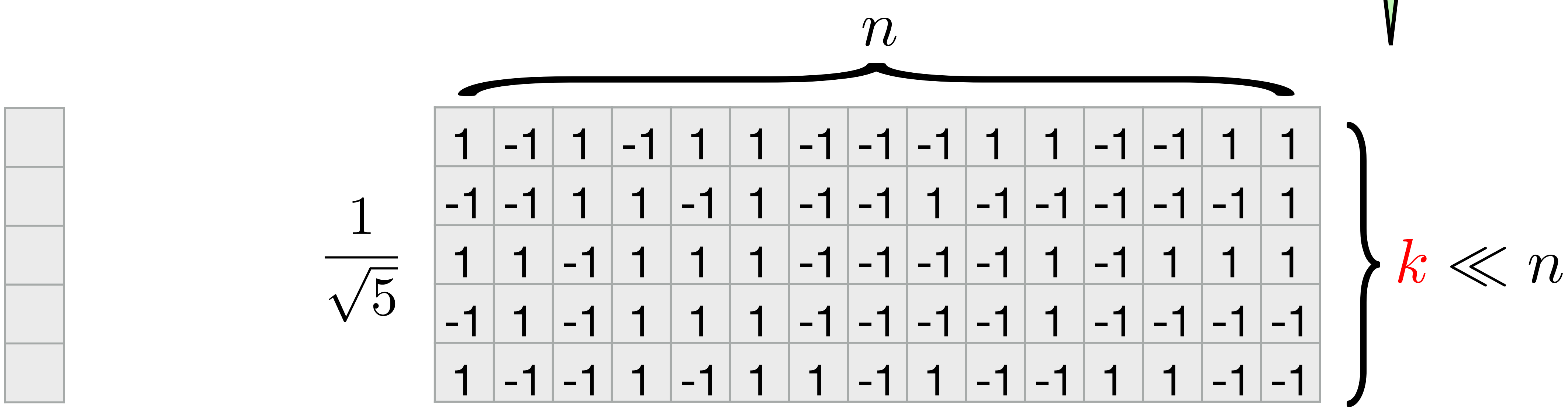

$$\frac{1}{\sqrt{5}}\overbrace{\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \end{bmatrix}}^{n}\left.\vphantom{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}\right\} k \ll n$$

Sketching vector $\Pi f^{(t)}$

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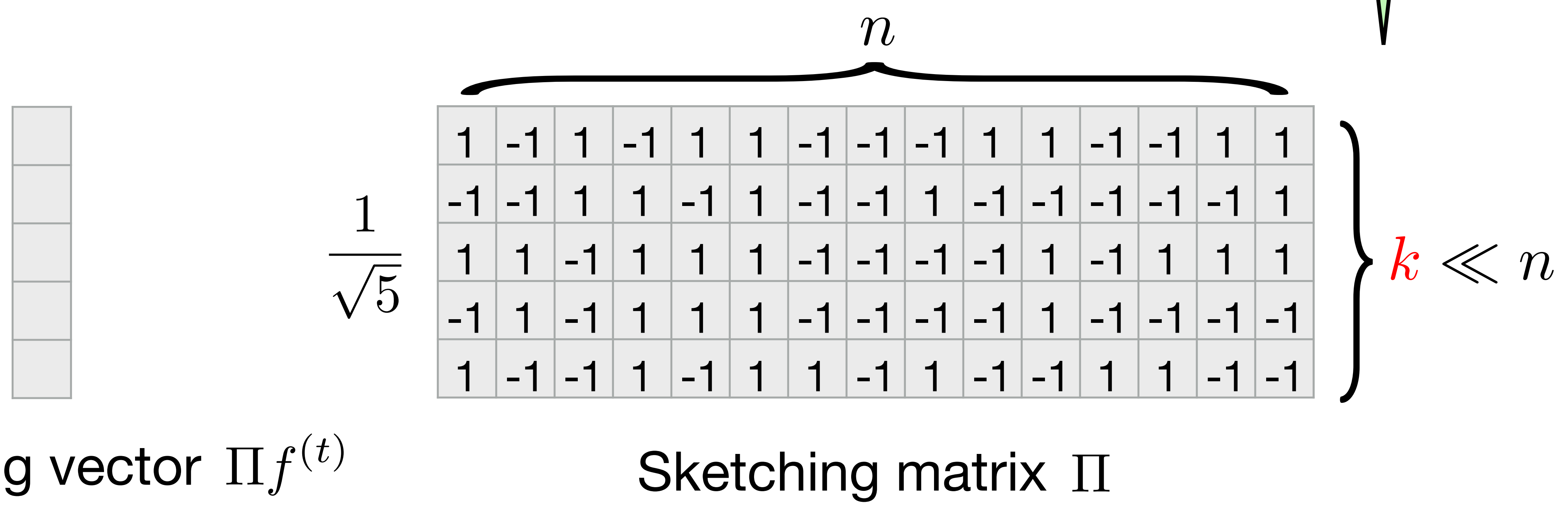

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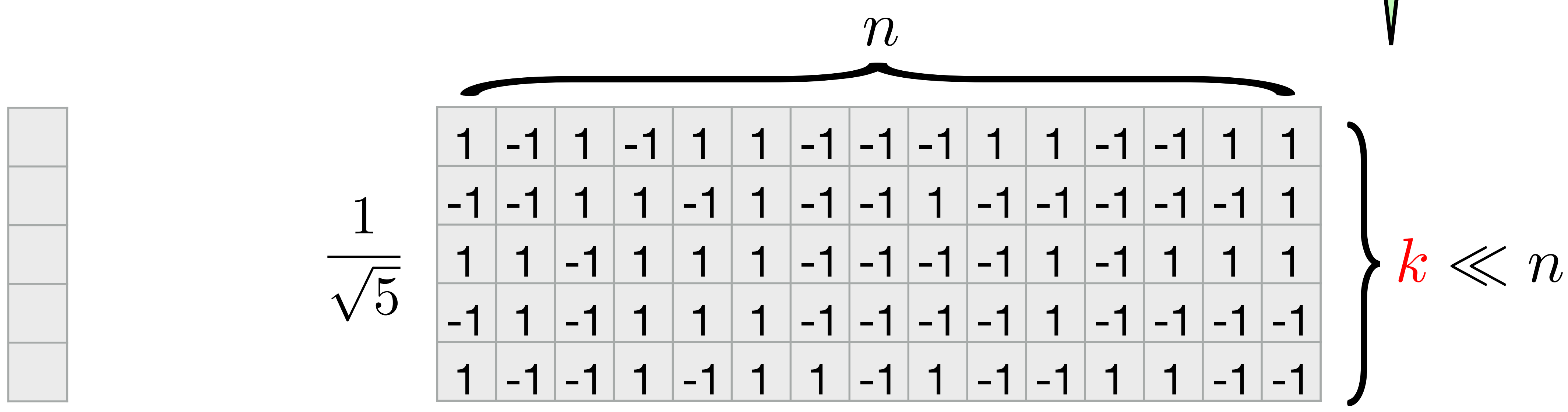

1	-1	1	-1	1	1	-1	-1	-1	1	1	-1	-1	1	1
-1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	-1	1
1	1	-1	1	1	1	-1	-1	-1	-1	1	-1	1	1	1
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- **AMS sketch:** $k = O(\epsilon^{-2})$ for one-shot [Alon-Matias-Szegedy 96] and for weak tracking [Braverman-Chestnut-Ivkin-Nelson-Wang-Woodruff 17].

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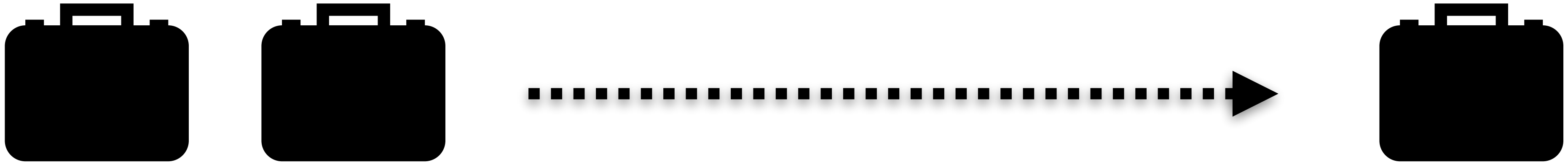

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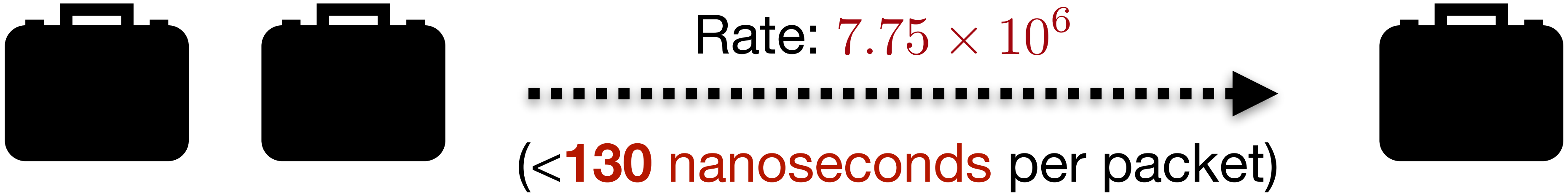

Rate: 7.75×10^6

(<**130** nanoseconds per packet)

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Q: Is AMS Sketch optimal in update time complexity?

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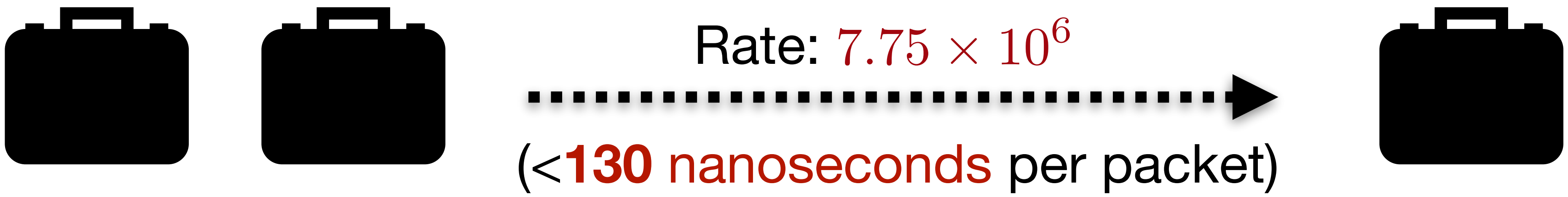

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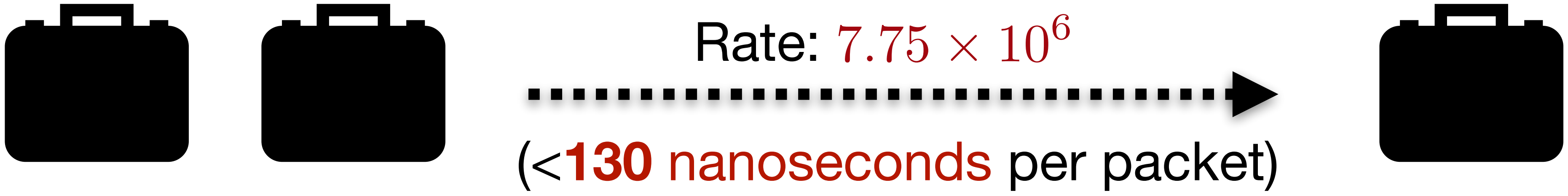

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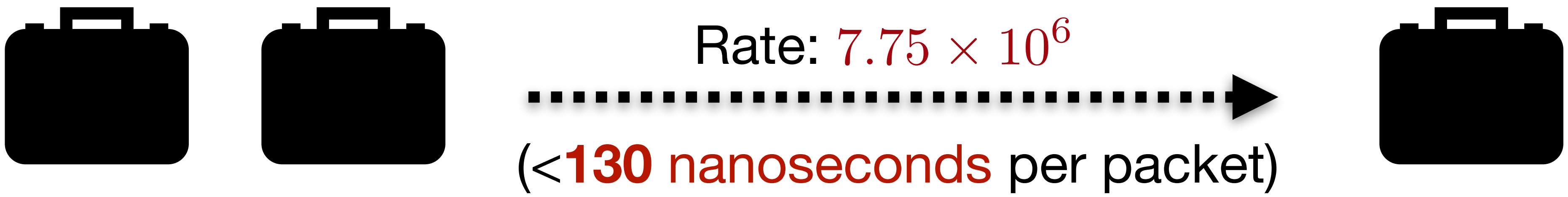

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There is an $O(1)$ time algorithm provides $(\epsilon, 0.1)$-weak tracking.

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\hline
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-1} & 0 & 0 & 0 \\
\hline
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$\Pi =$

0	0	0	0	0	**1**	0	0	0	0	**1**	**-1**	0	0	0
0	**-1**	**1**	0	**-1**	0	0	0	0	0	0	0	0	**-1**	0
0	0	0	0	0	0	0	**-1**	**-1**	**-1**	0	0	**1**	0	0
-1	0	0	**1**	0	0	**-1**	0	0	0	0	0	0	0	0
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  - Using $O\left(\epsilon^{-2}\delta^{-1}m\right)$ rows (or $O\left(\epsilon^{-2}\delta^{-1}\log m\right)$ rows after median trick).

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We can get rid of the m dependency!

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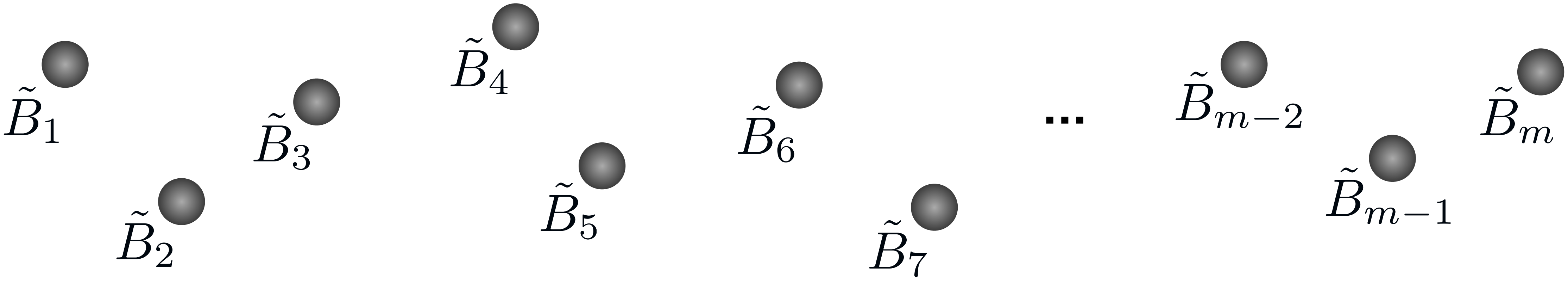

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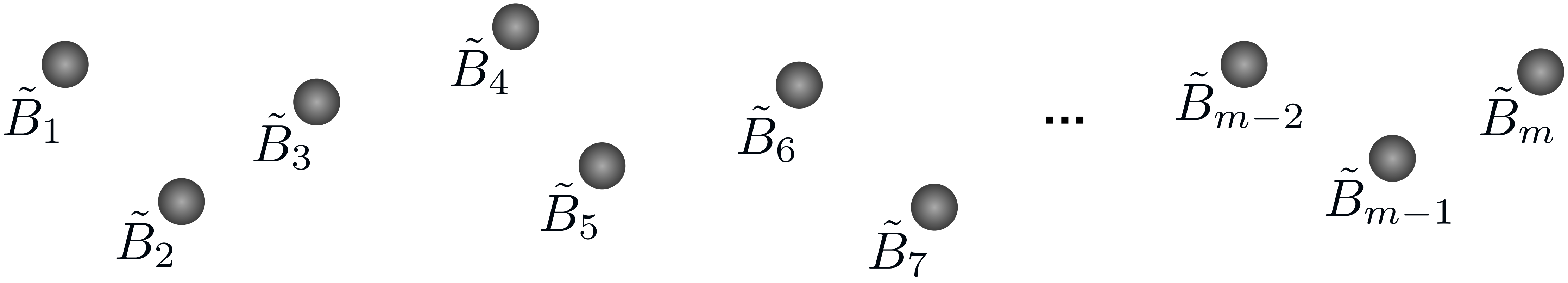

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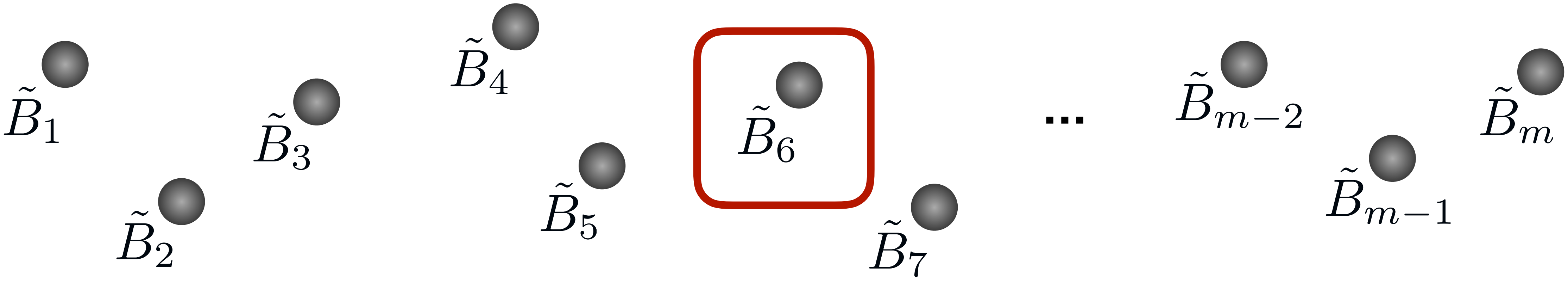

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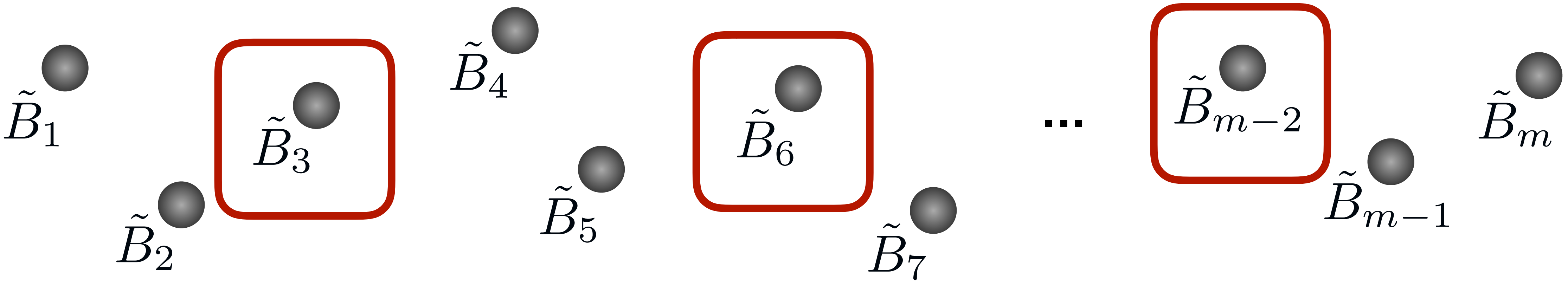

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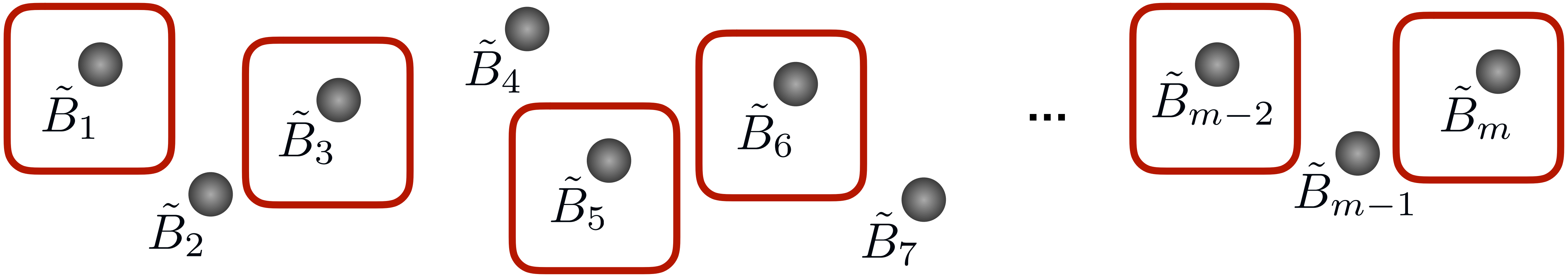

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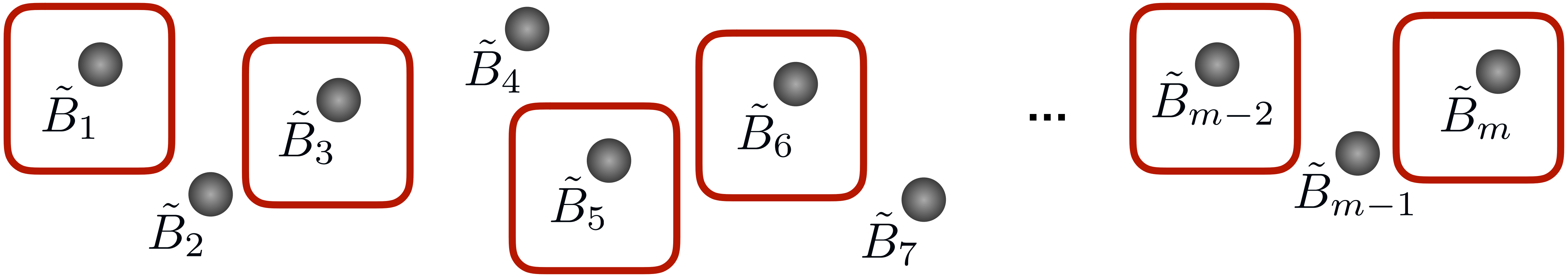

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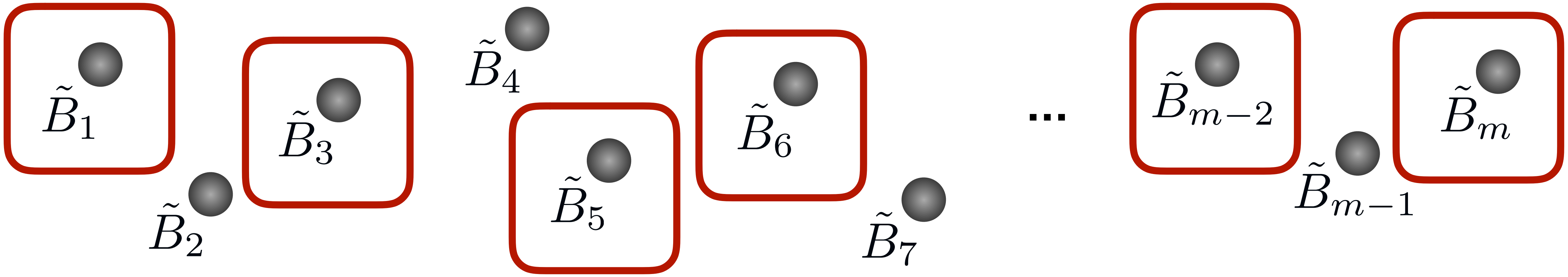

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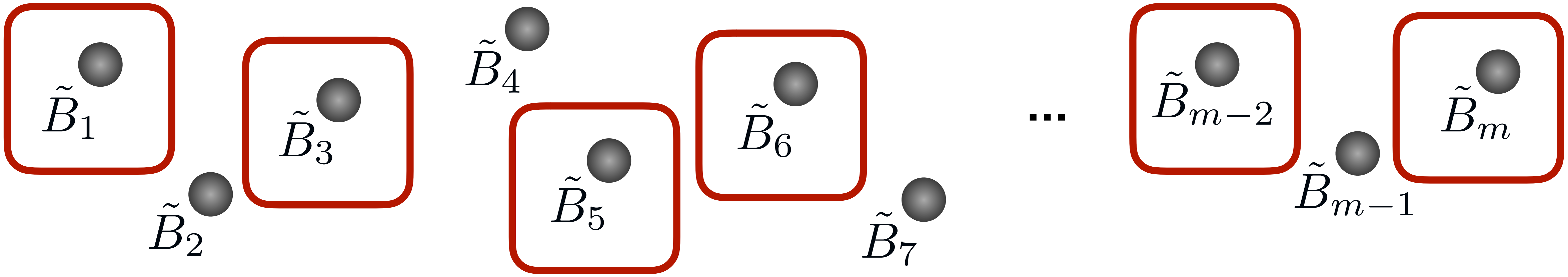

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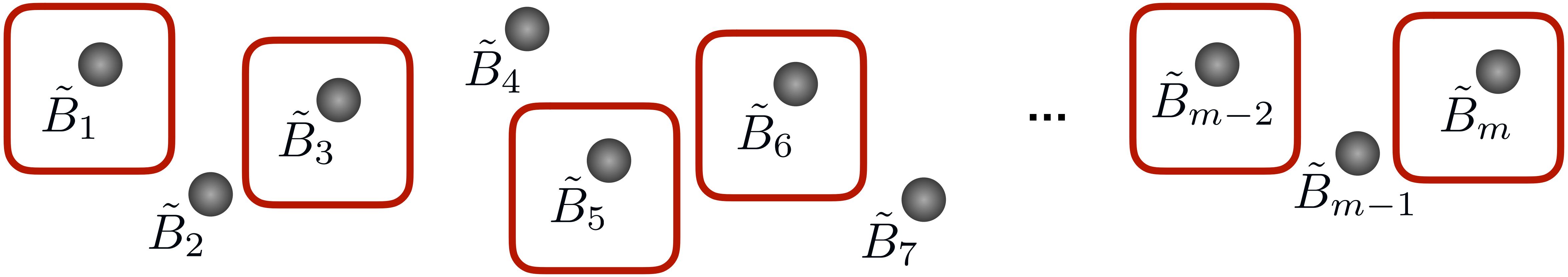

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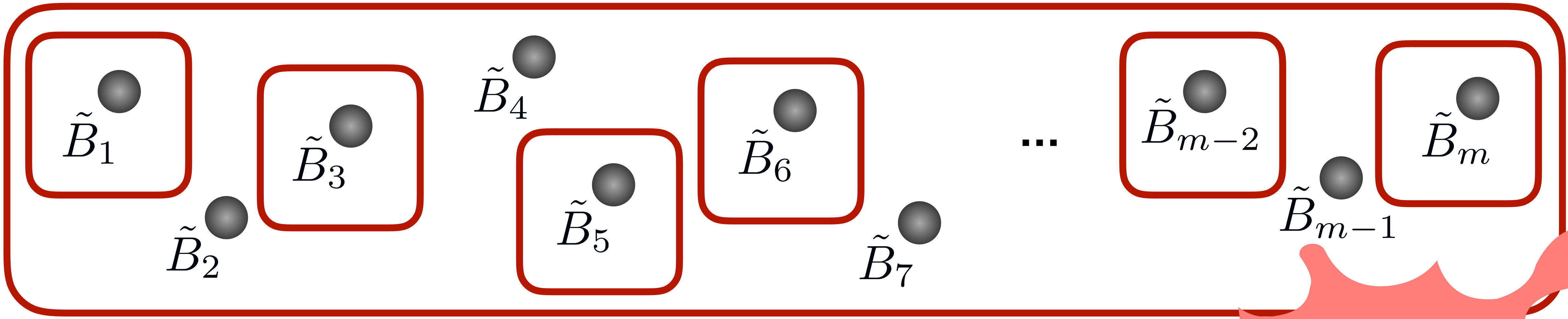

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Ask me offline for more details!

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