Hardness vs Randomness for Bounded Depth Arithmetic Circuits

Chi-Ning Chou Mrinal Kumar Noam Solomon

Harvard University

Outline

- Arithmetic circuits and algebraic complexity classes
- Polynomial identity testing (PIT)
- Hardness vs Randomness for arithmetic circuits
- Polynomial factorization
- Open problems

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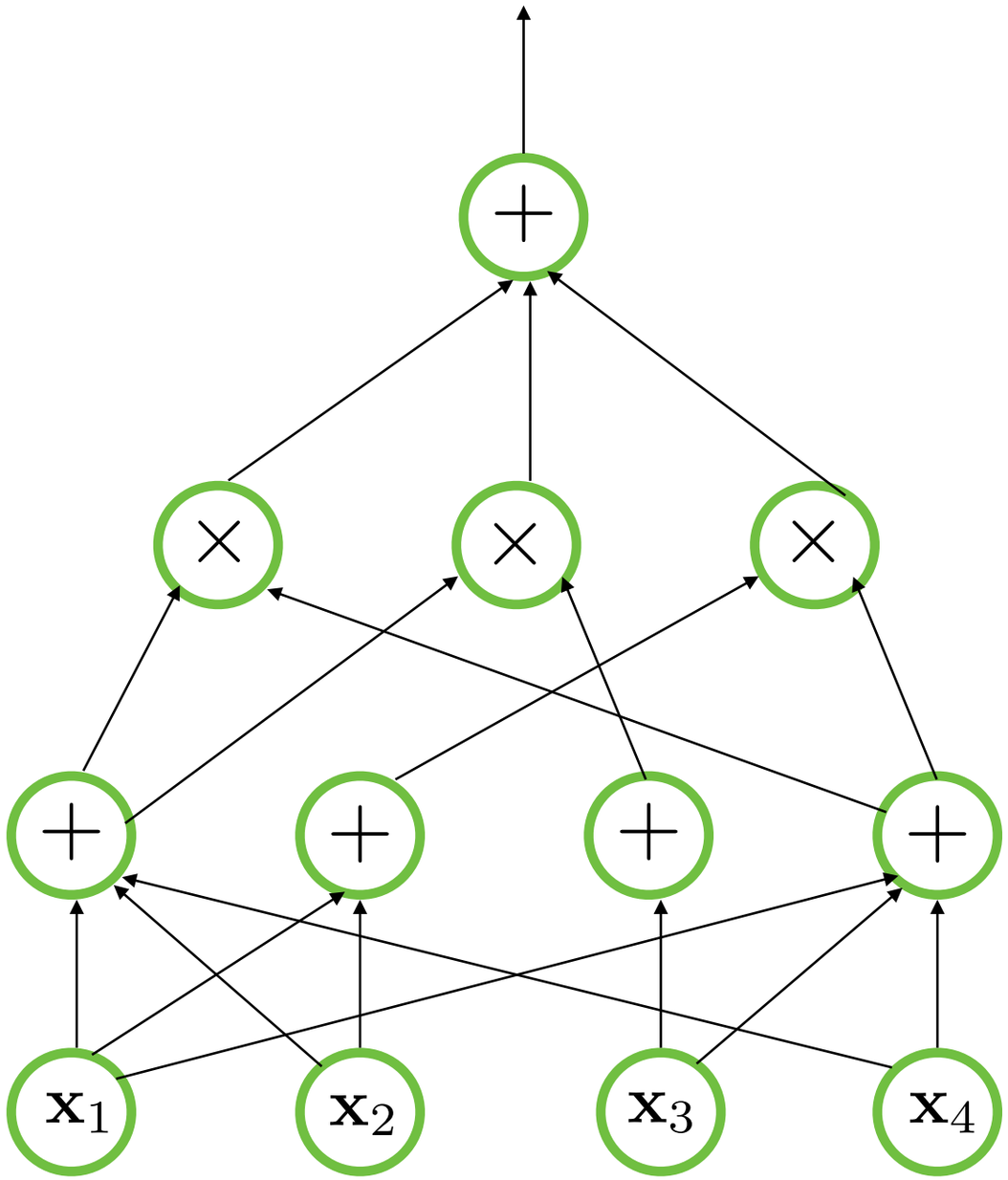

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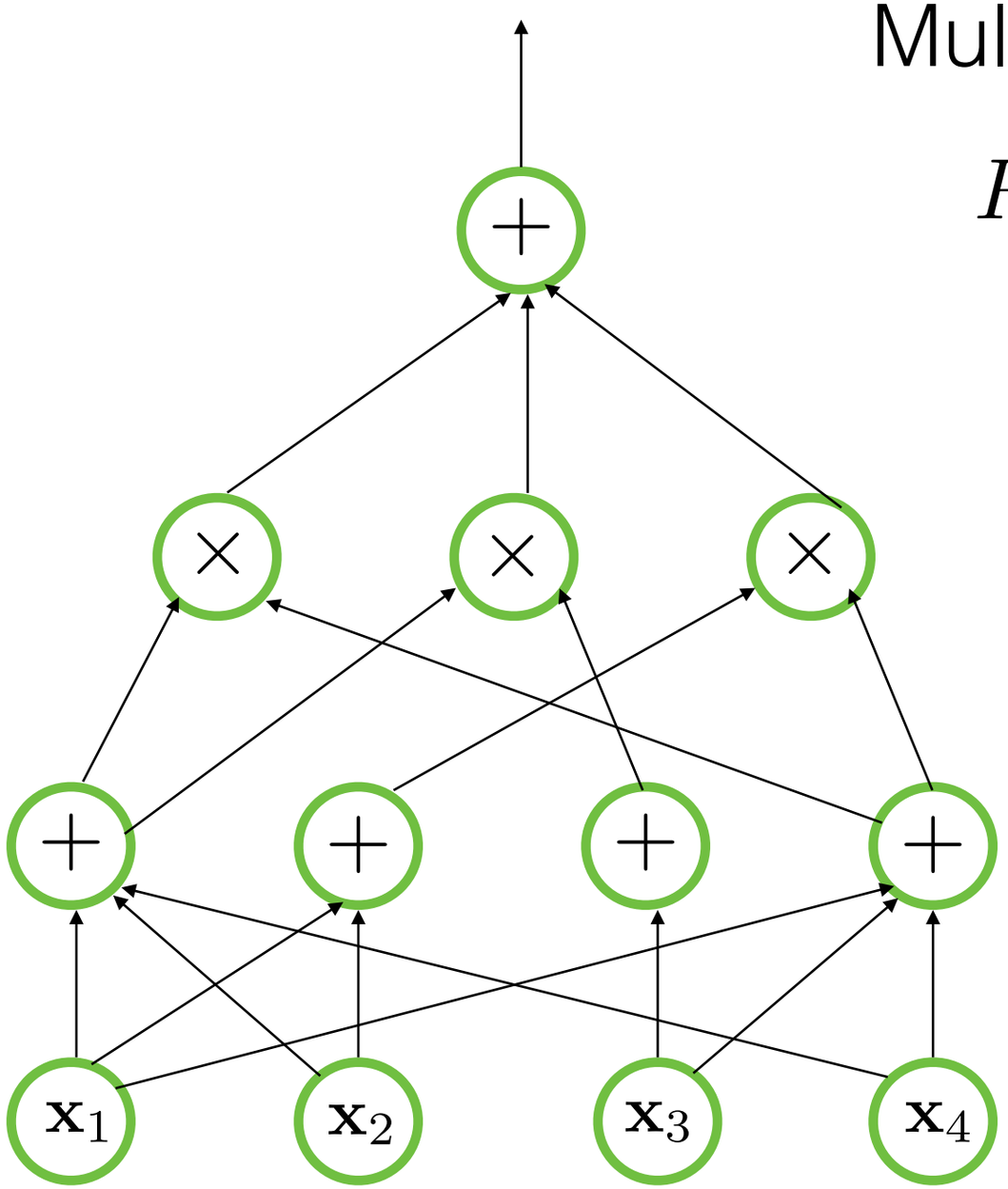

Multivariate polynomial

$$P \in \mathbb{F}[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$$

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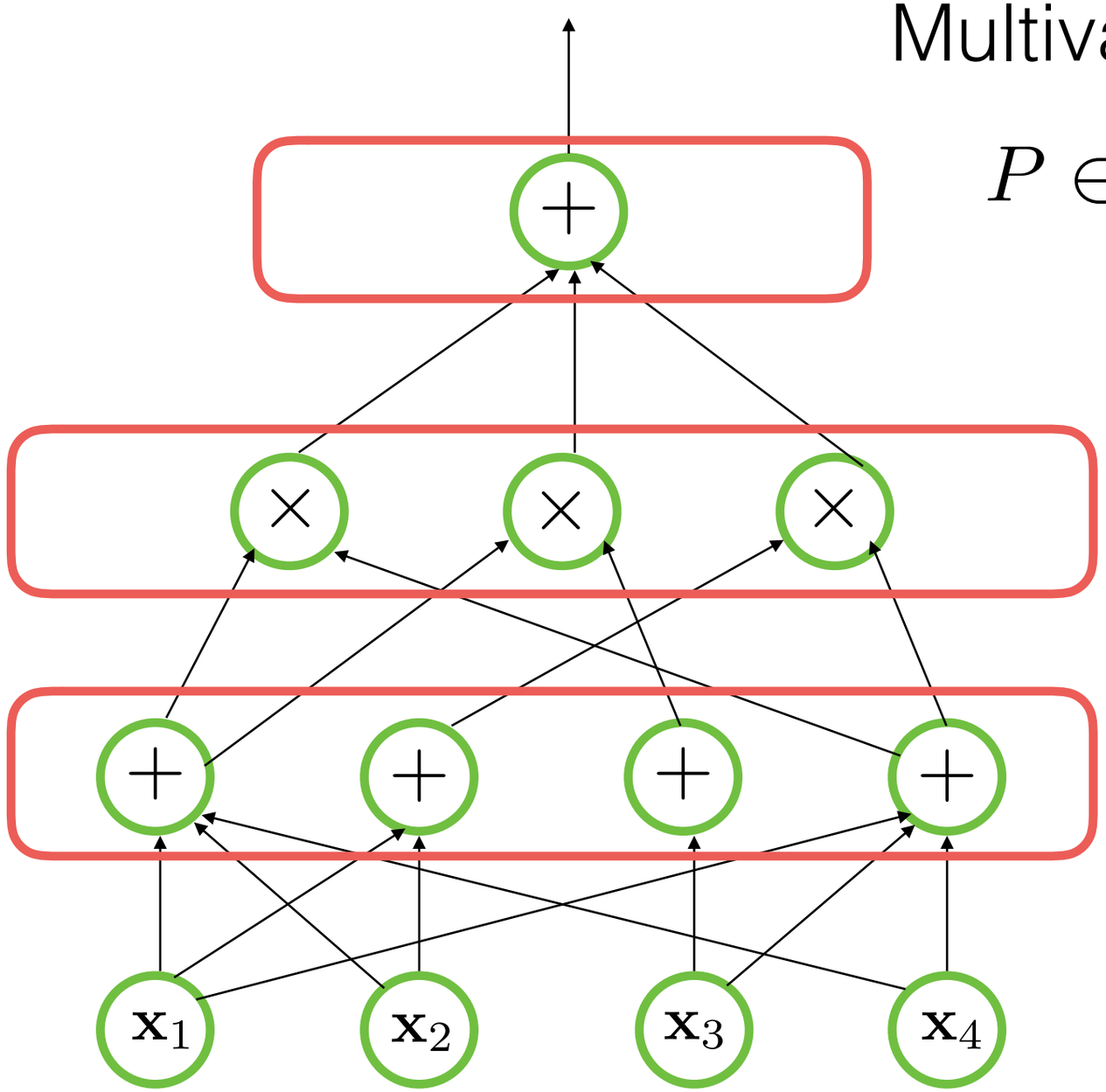

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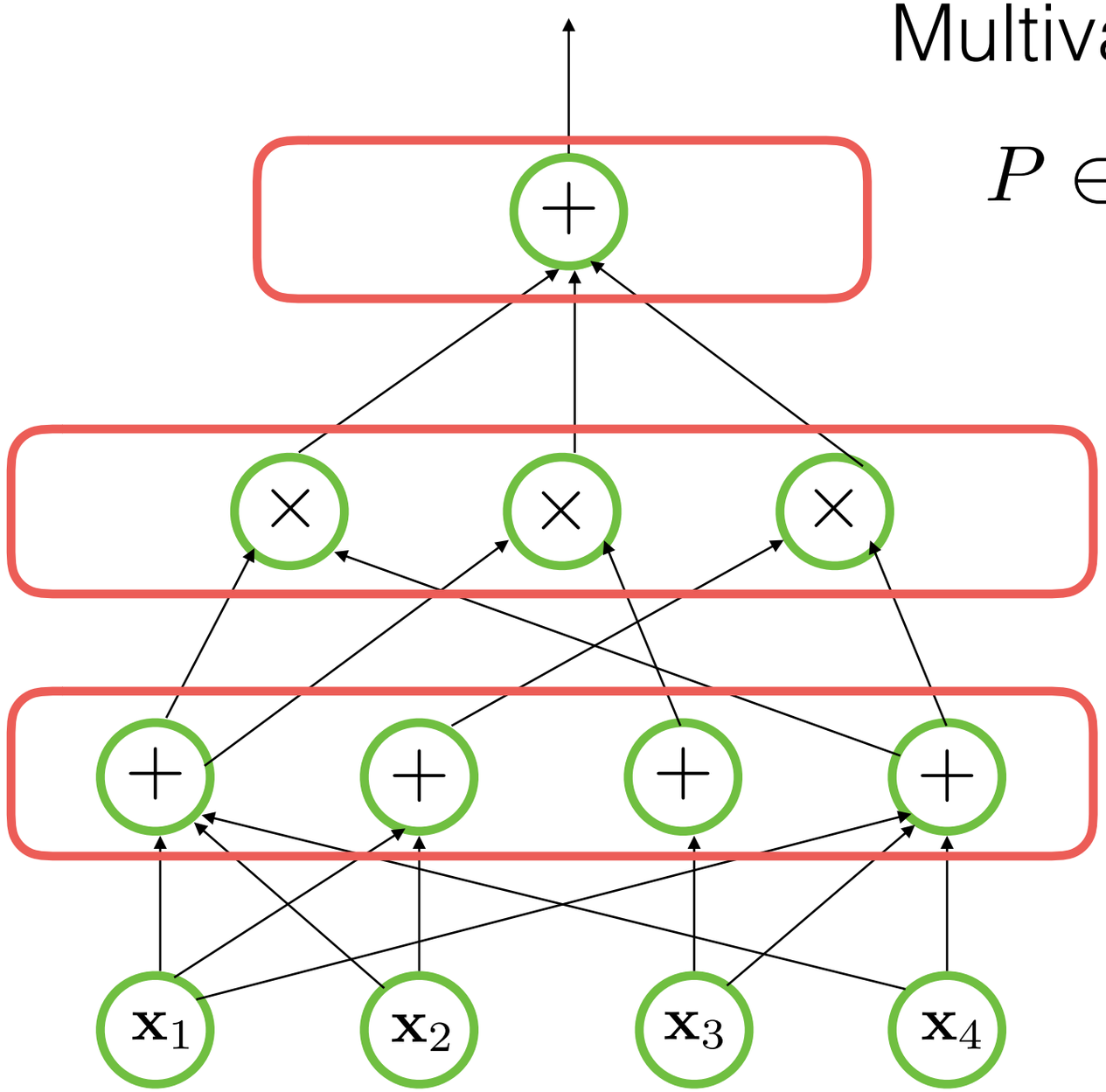

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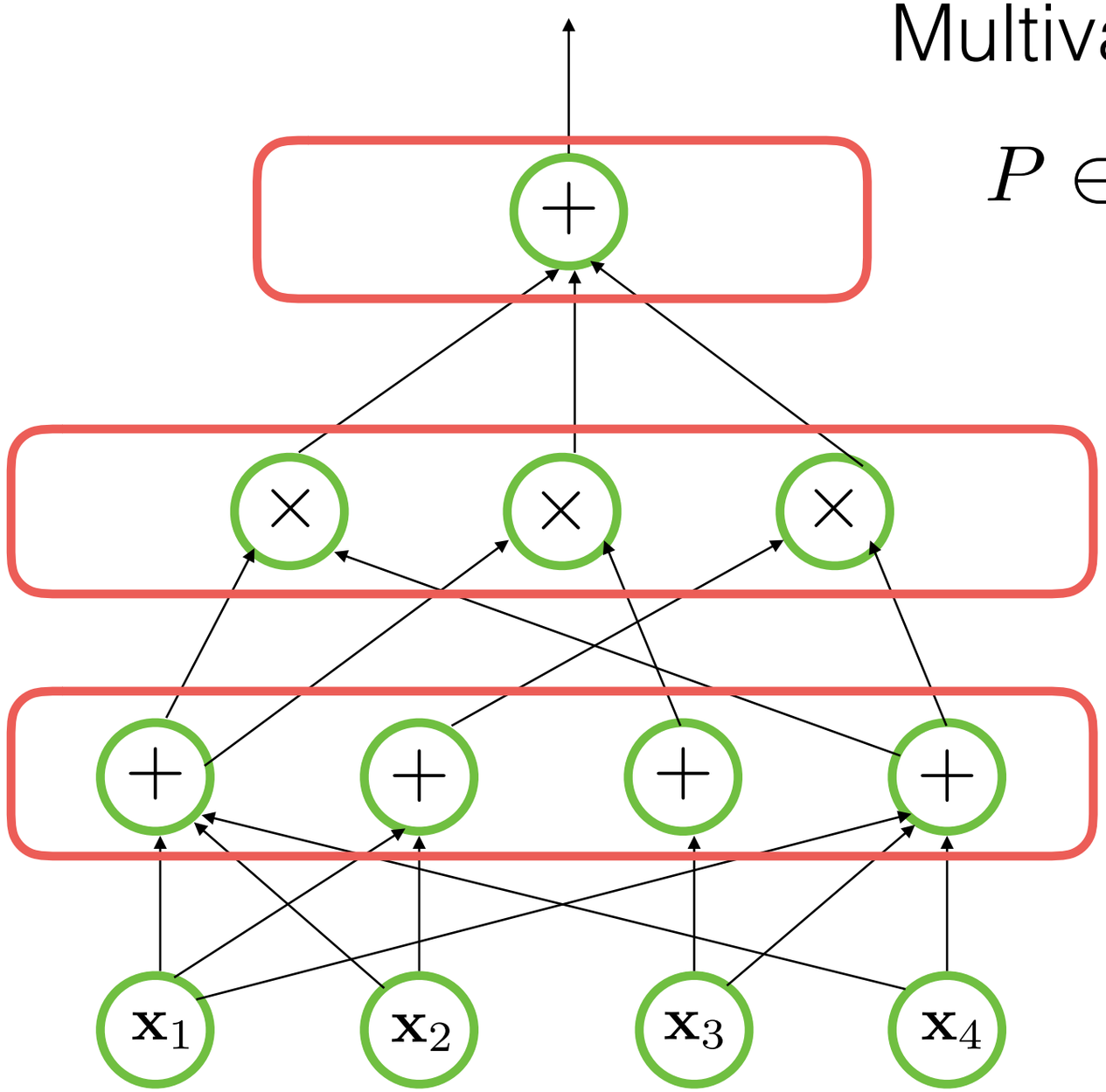

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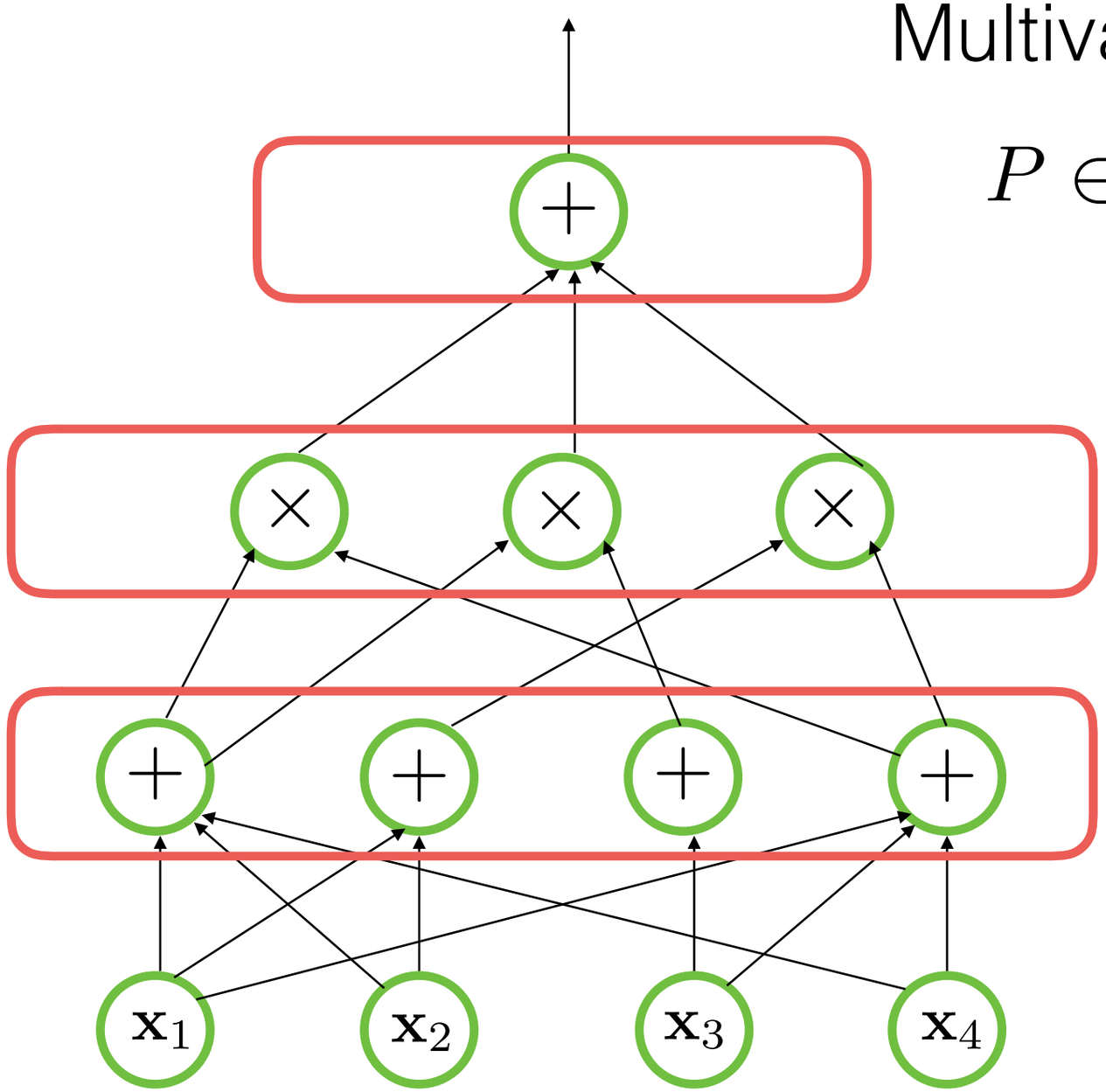

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*Assume $\mathbb{F} = \mathbb{Q}$

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- Many more such as VF, VBP, VNP…

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- [NW'95, GKKS'14, FLMS'14, KS'14] **Exponential** lower bounds for depth-3 and depth-4 circuits.

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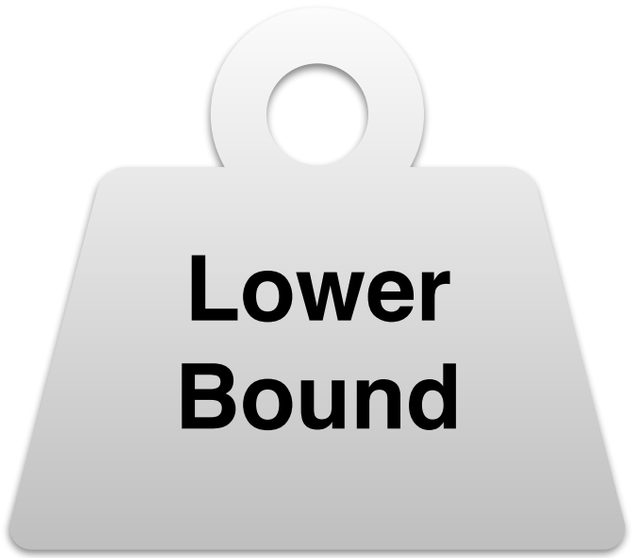

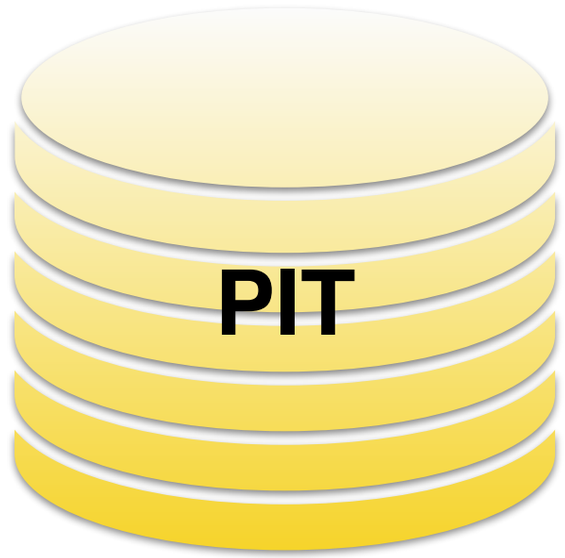

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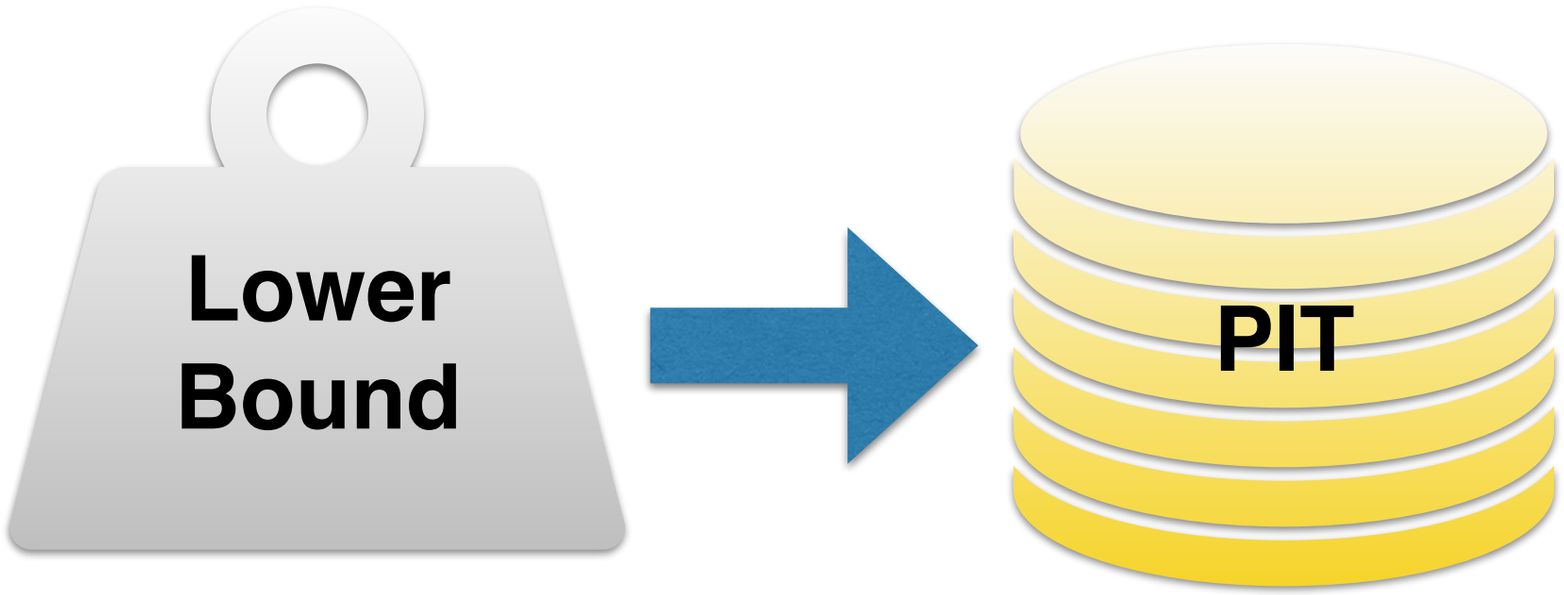

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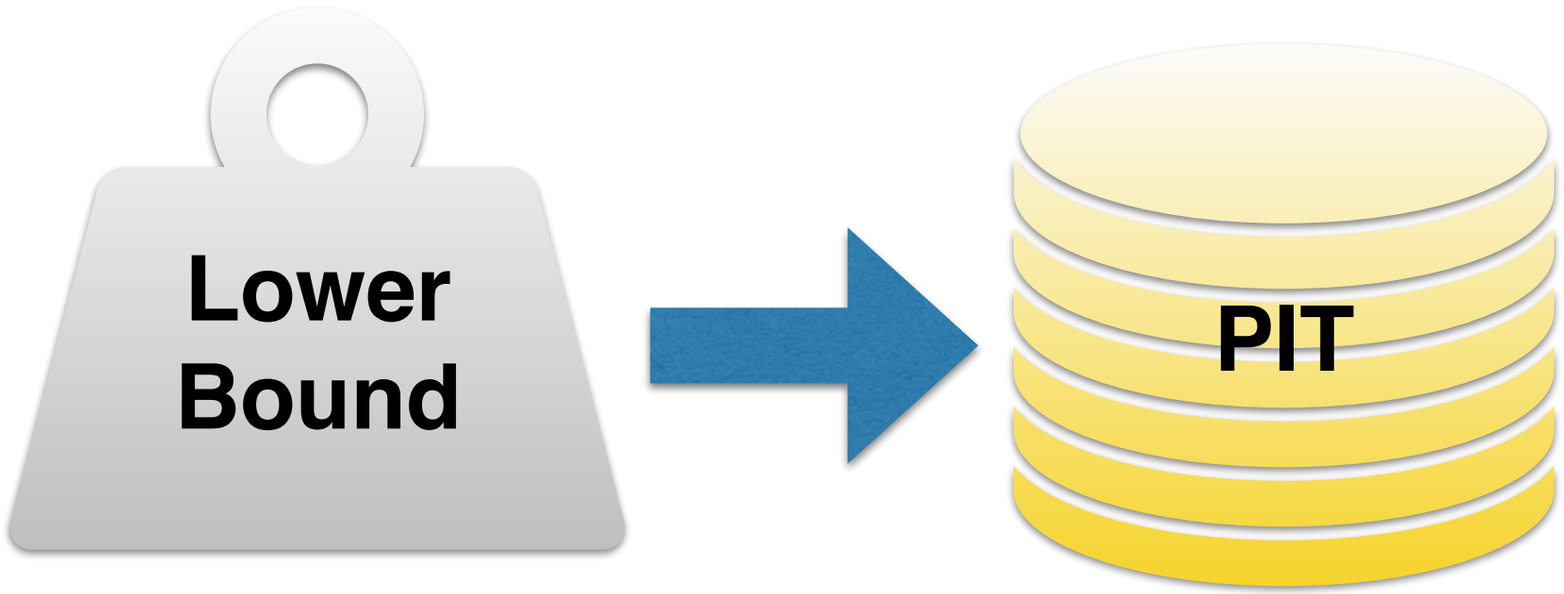

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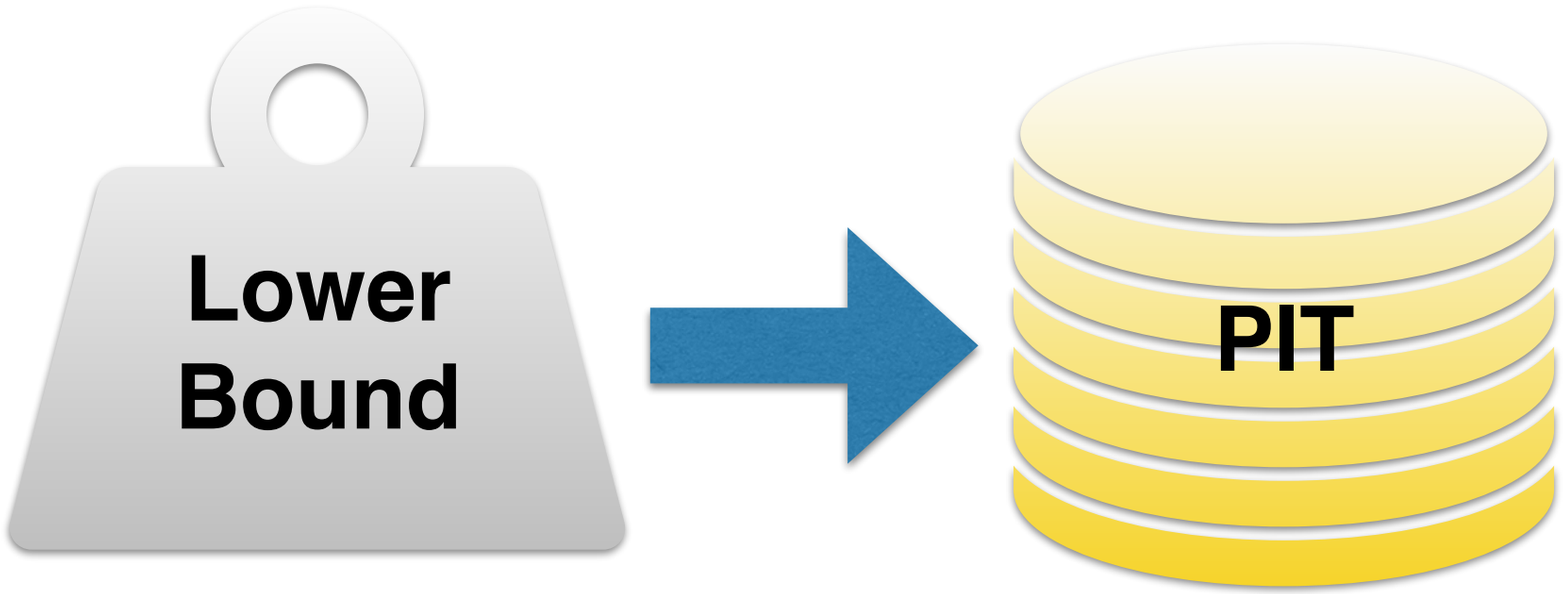

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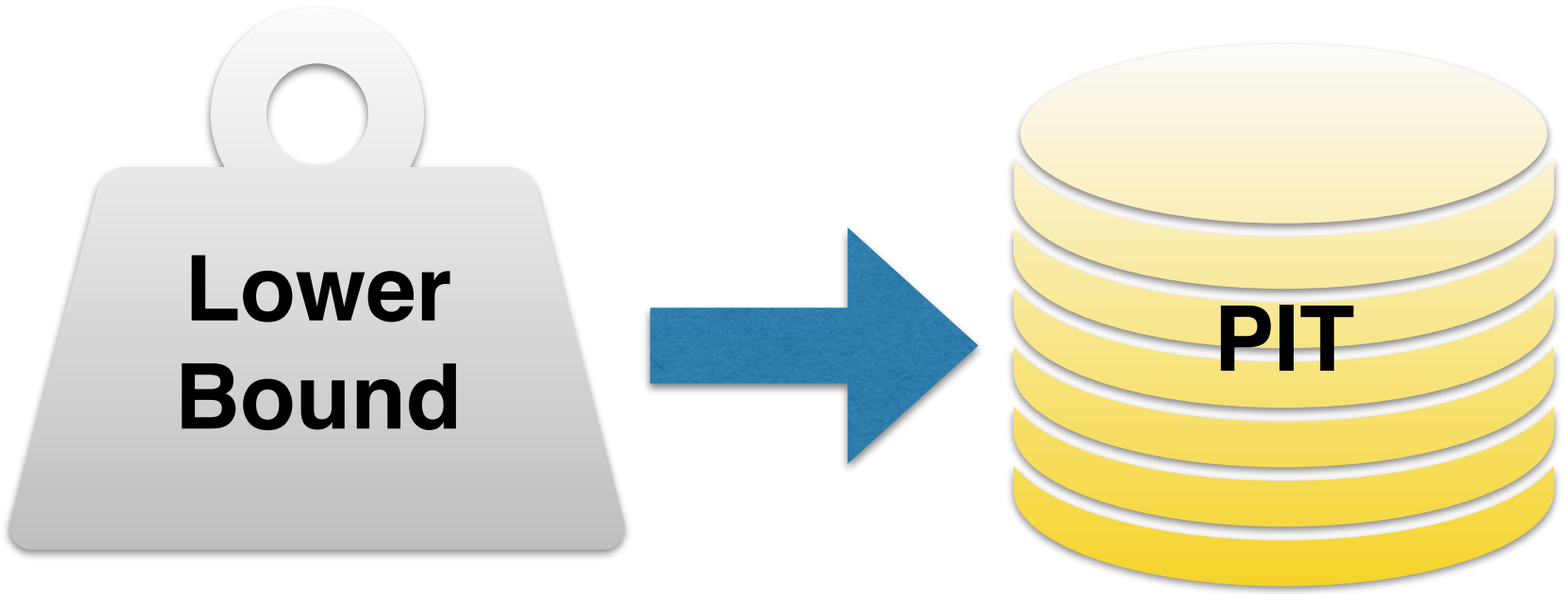

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Don't be bothered by the constant in depth!

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Compare with [Dvir-Shpilka-Yehudayoff'09]

	[DSY'09]	This work
Lower bound for $\text{Depth-}\Delta$		With degree $O(\log^2 n / \log^2 \log n)$
PIT for $\text{Depth-}\Delta - 5$	With bounded individual degree	

Hardness vs Randomness framework [KI'04, DSY'09]

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Nisan-Wigderson generator

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Schwartz-Zippel lemma

Brute-force to find hitting set in time $d^{O(\ell)}$

Hardness vs Randomness framework [KI'04, DSY'09]

Reduce to **factoring problem**!

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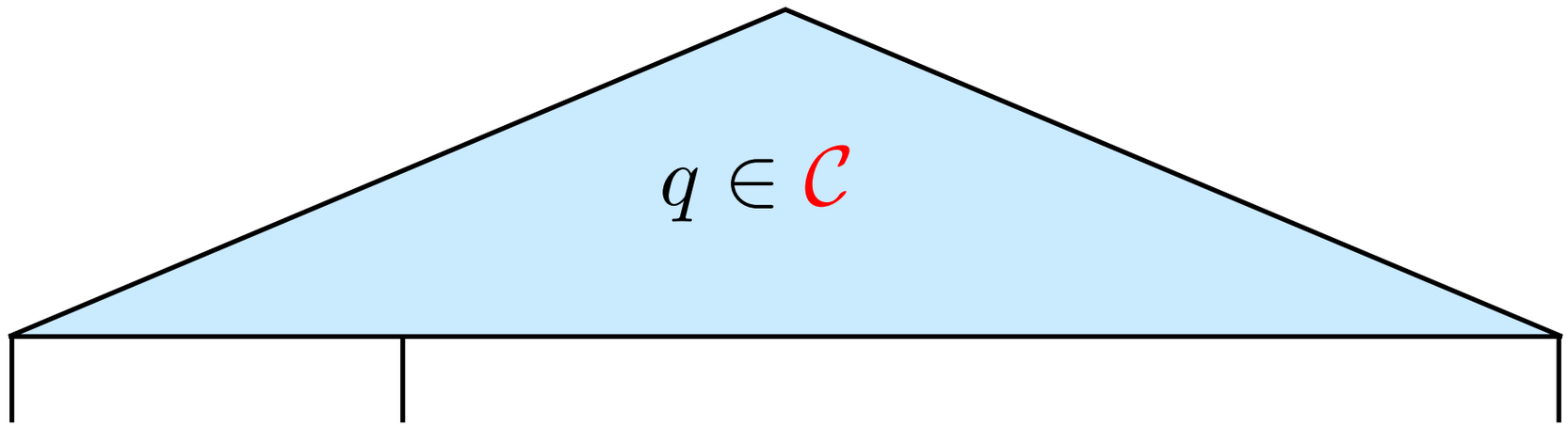
NW generator - reducing #variables



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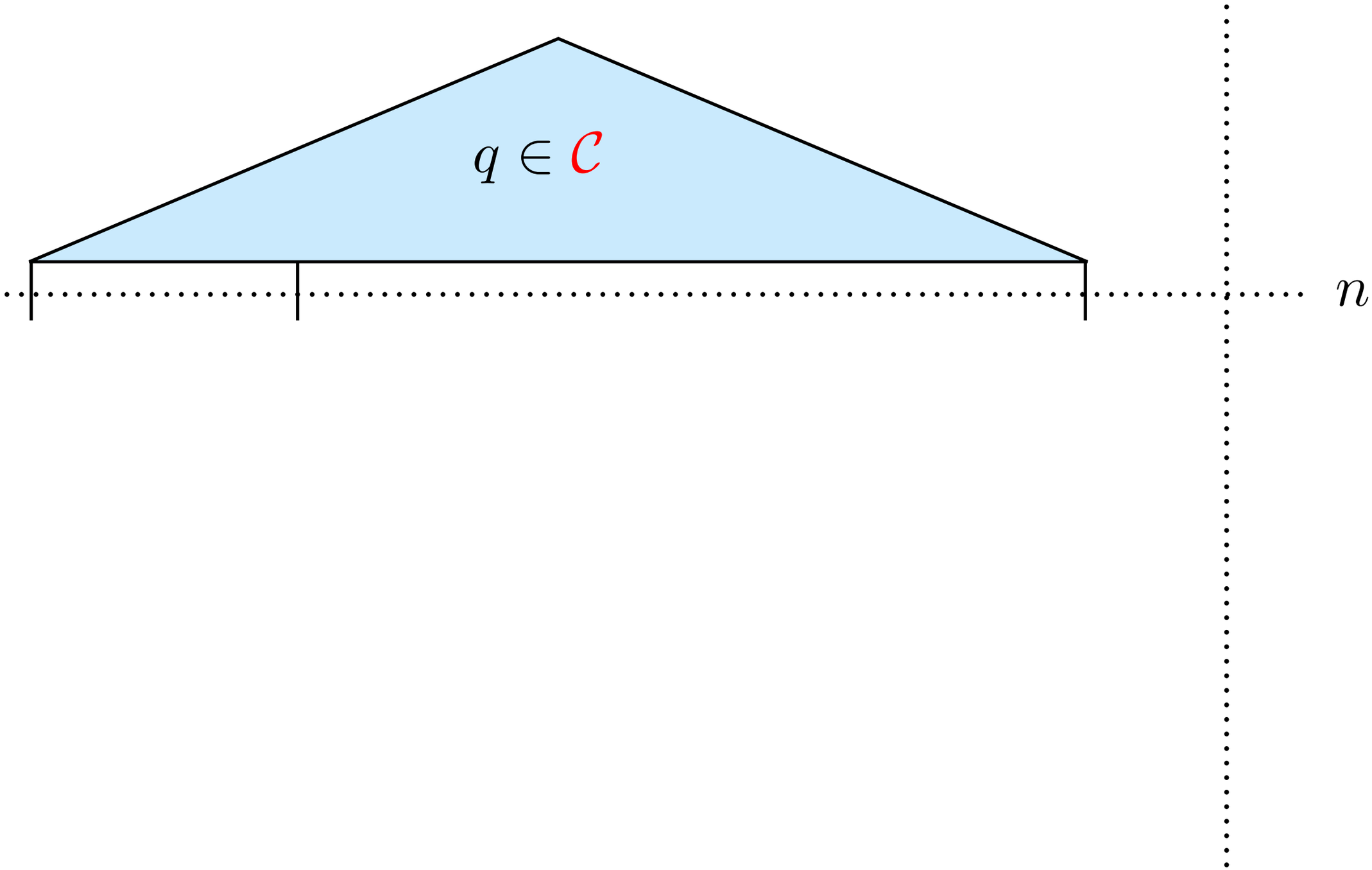

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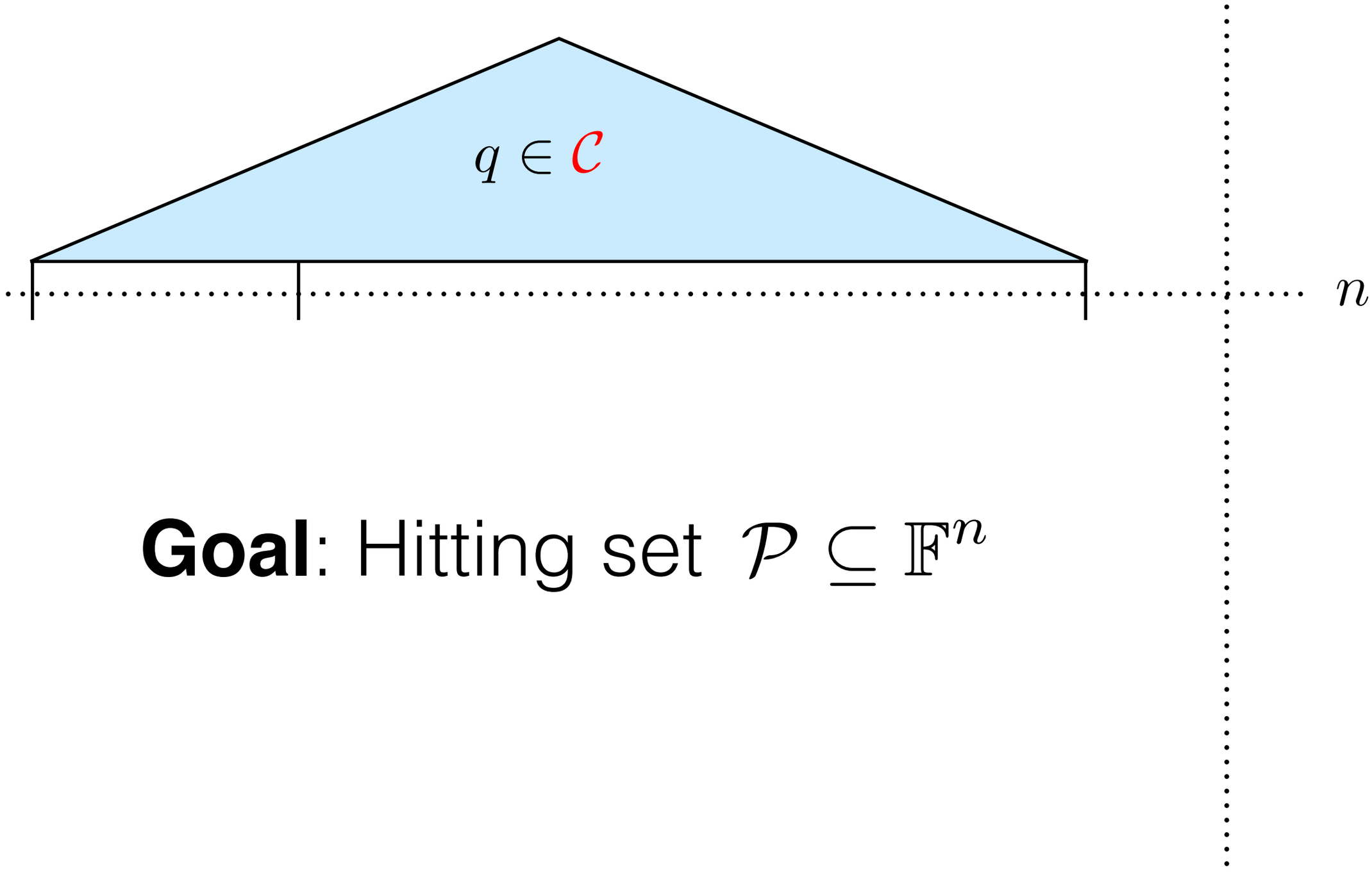

Goal: Hitting set $\mathcal{P} \subseteq \mathbb{F}^n$

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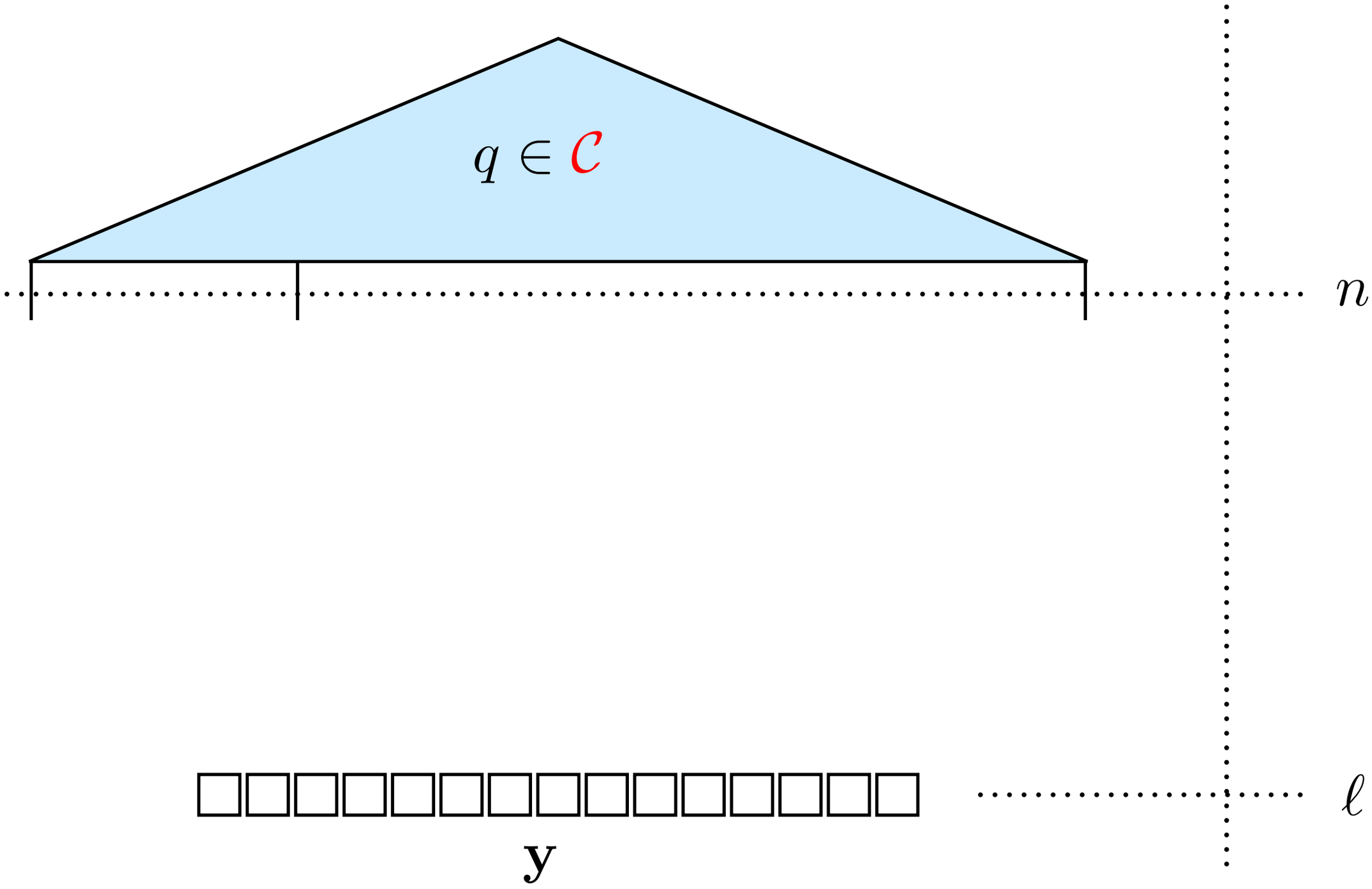

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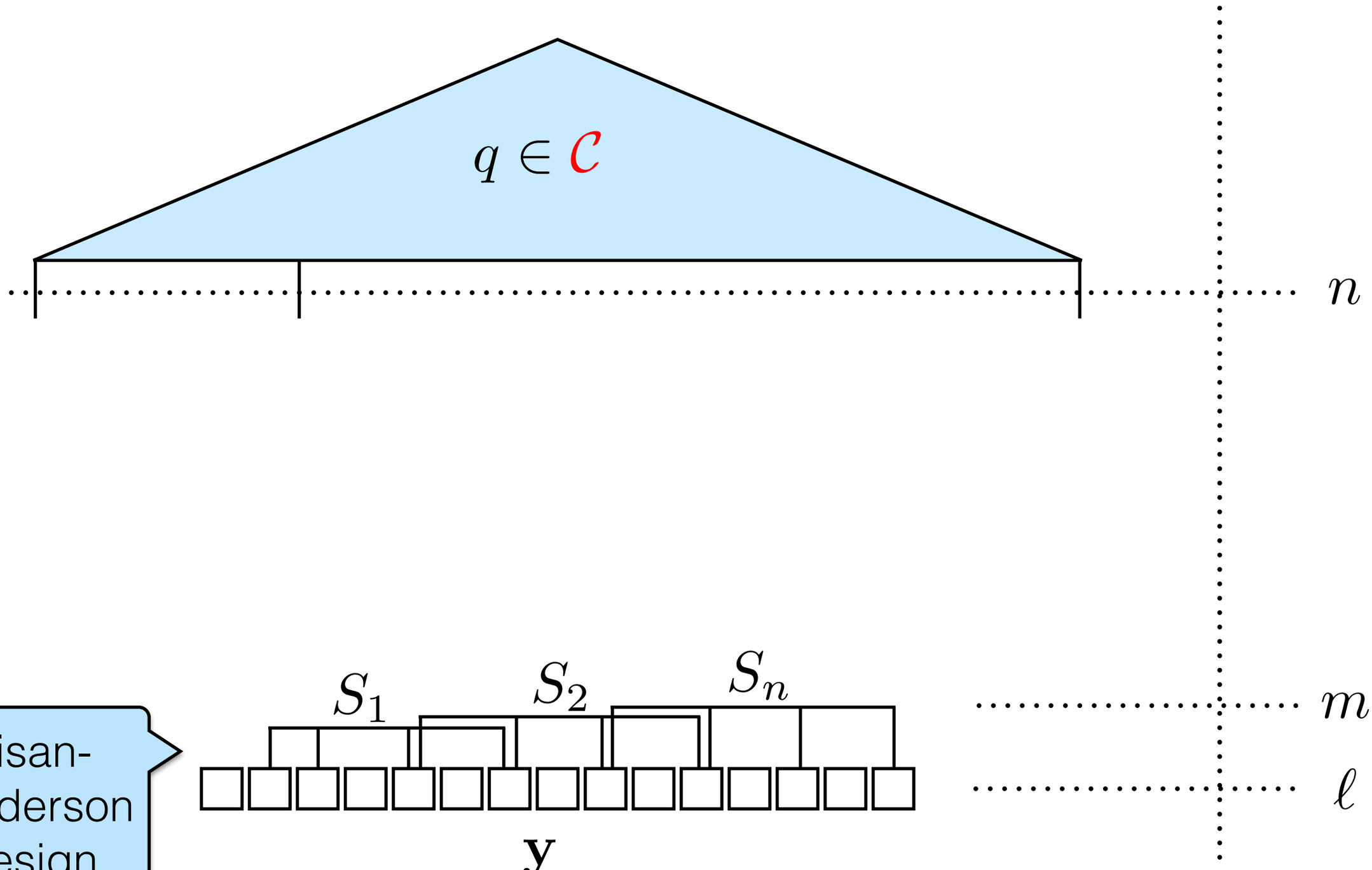

Nisan-Wigderson Design

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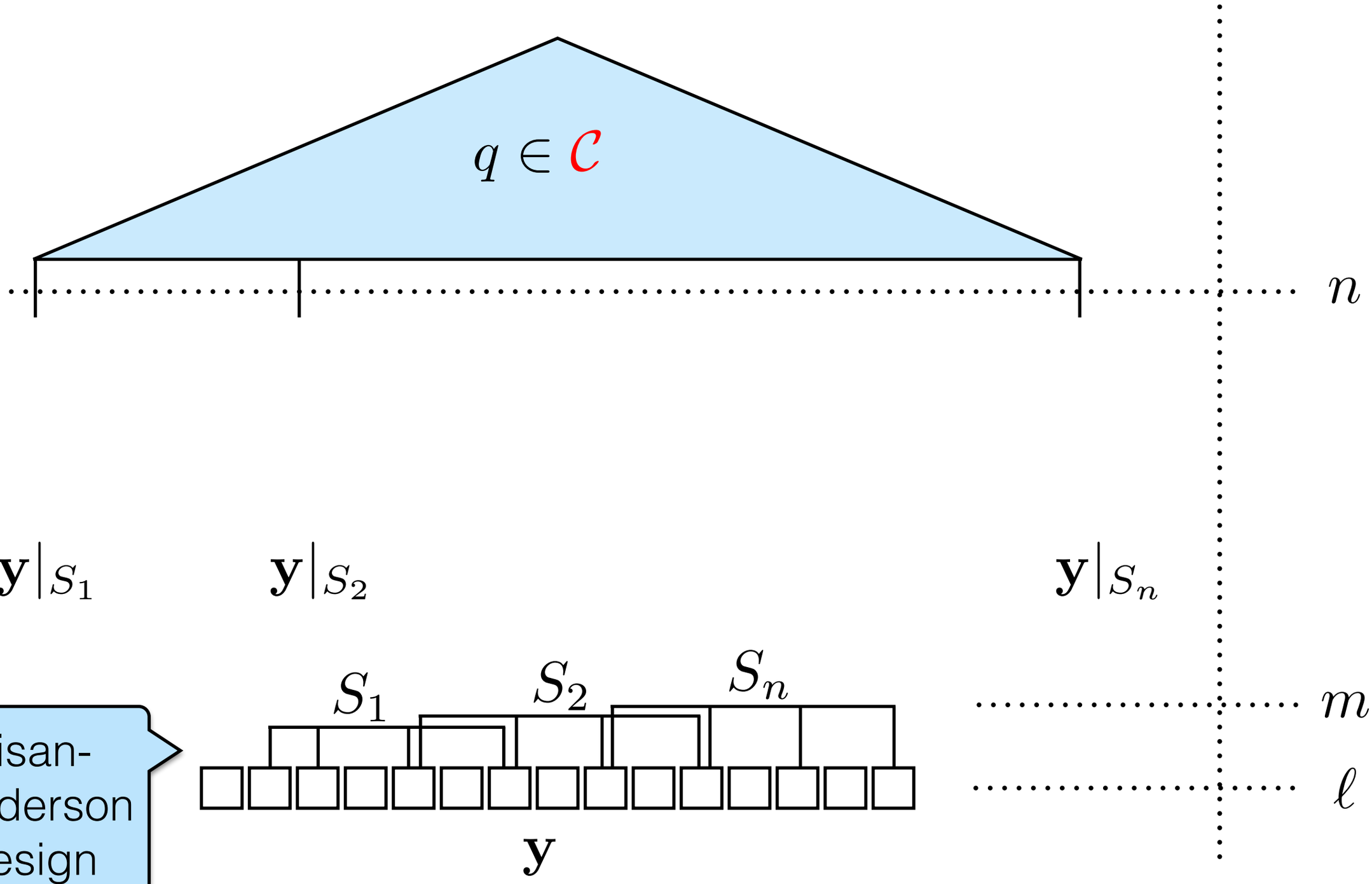

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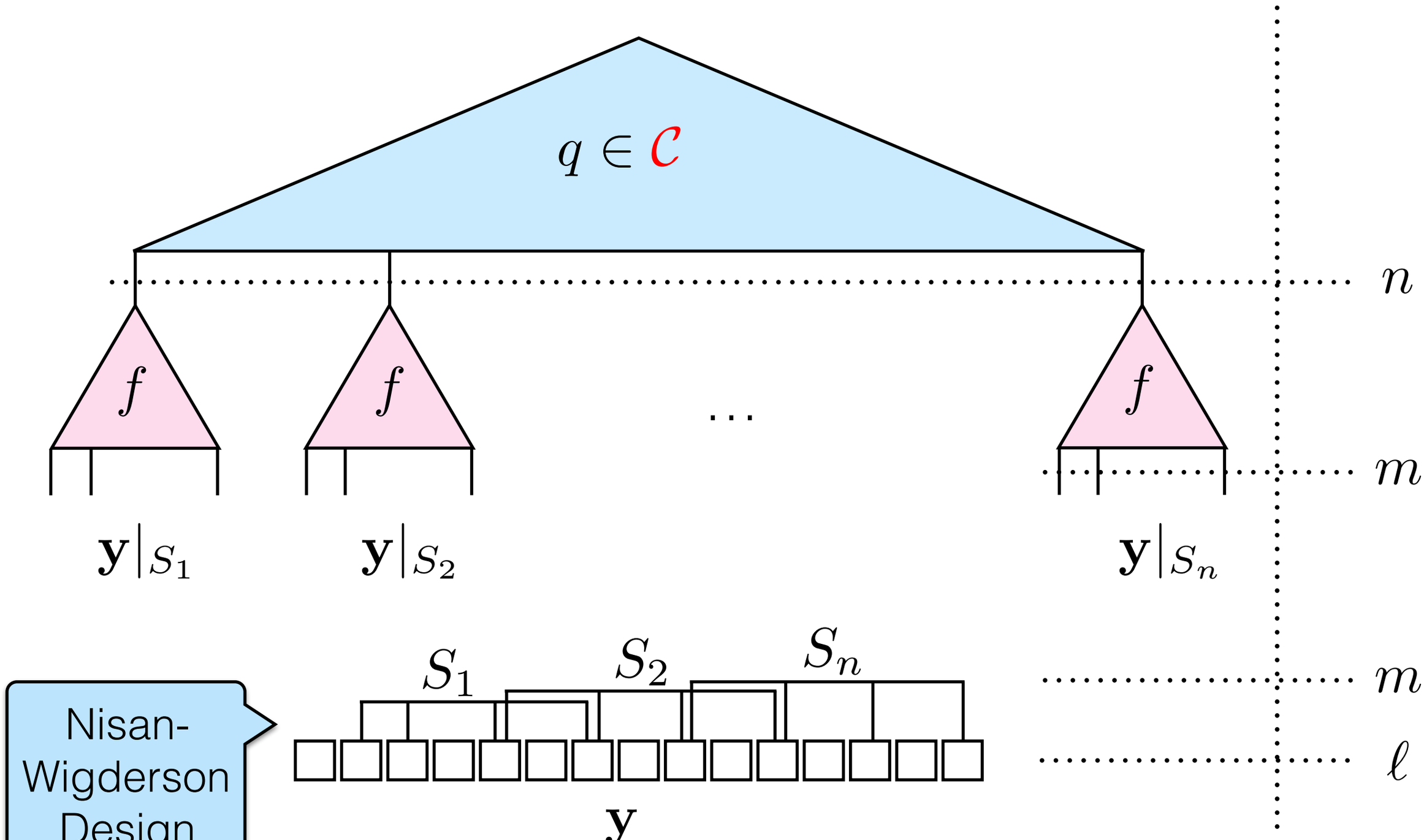

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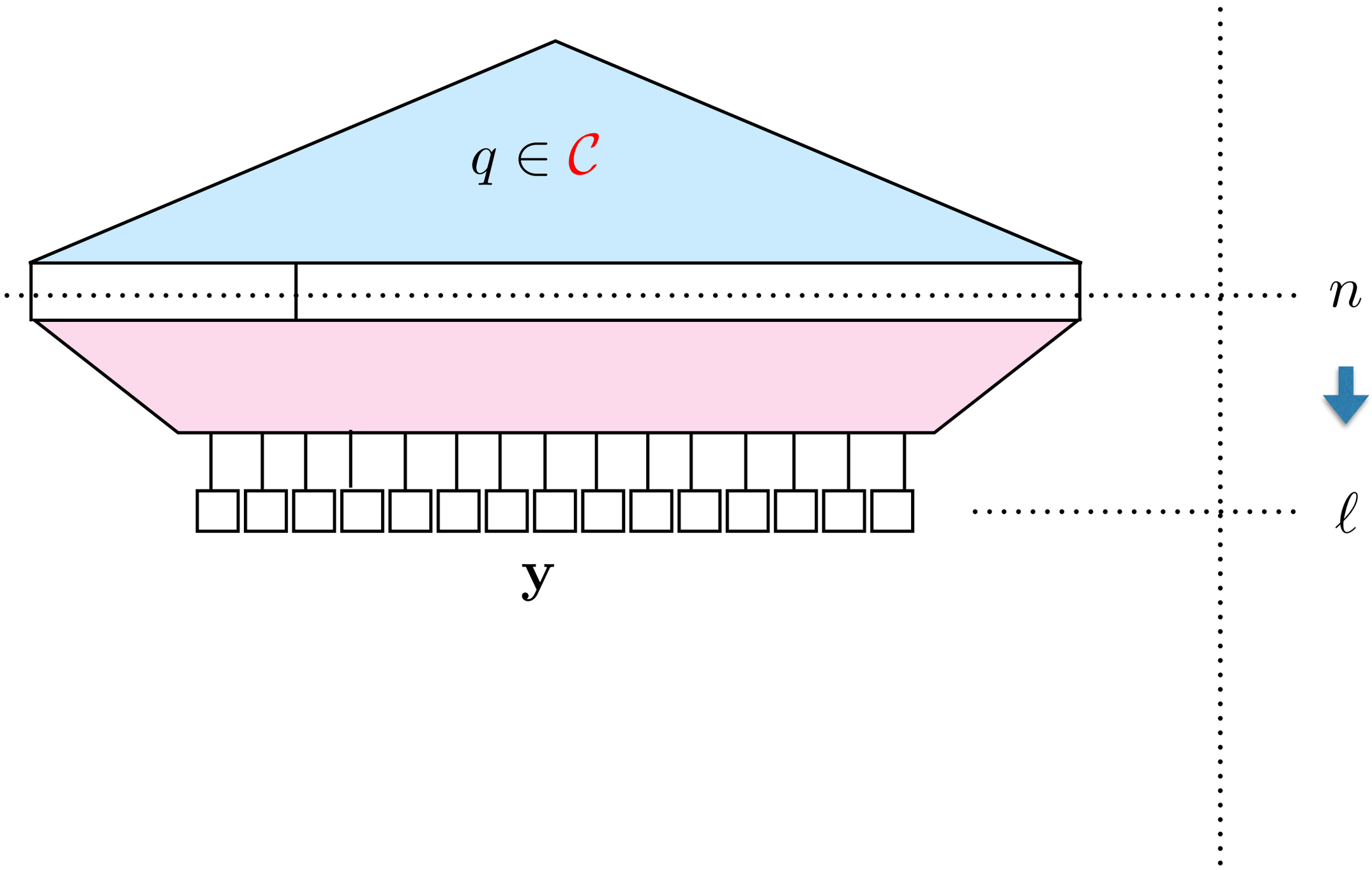

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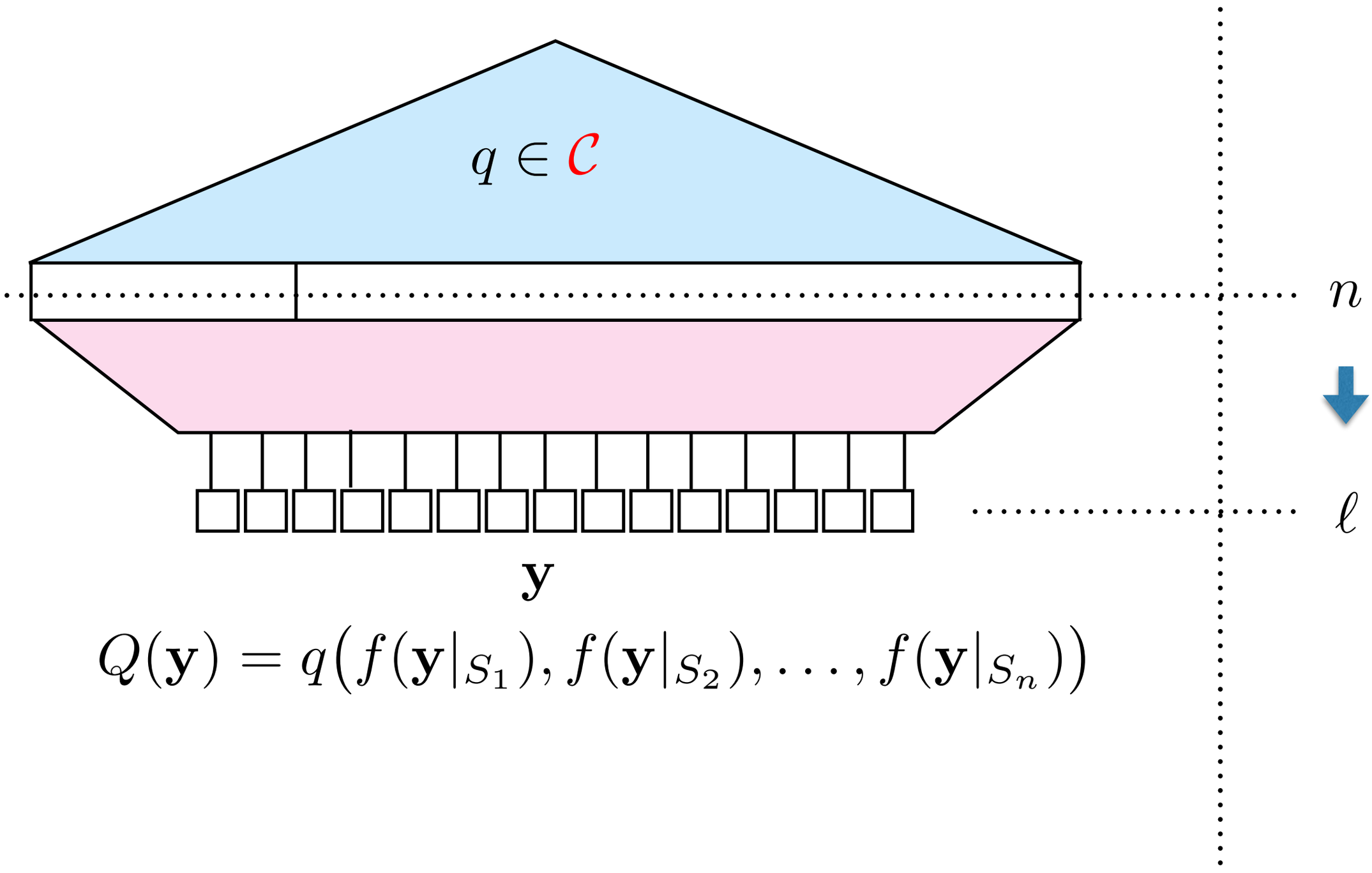

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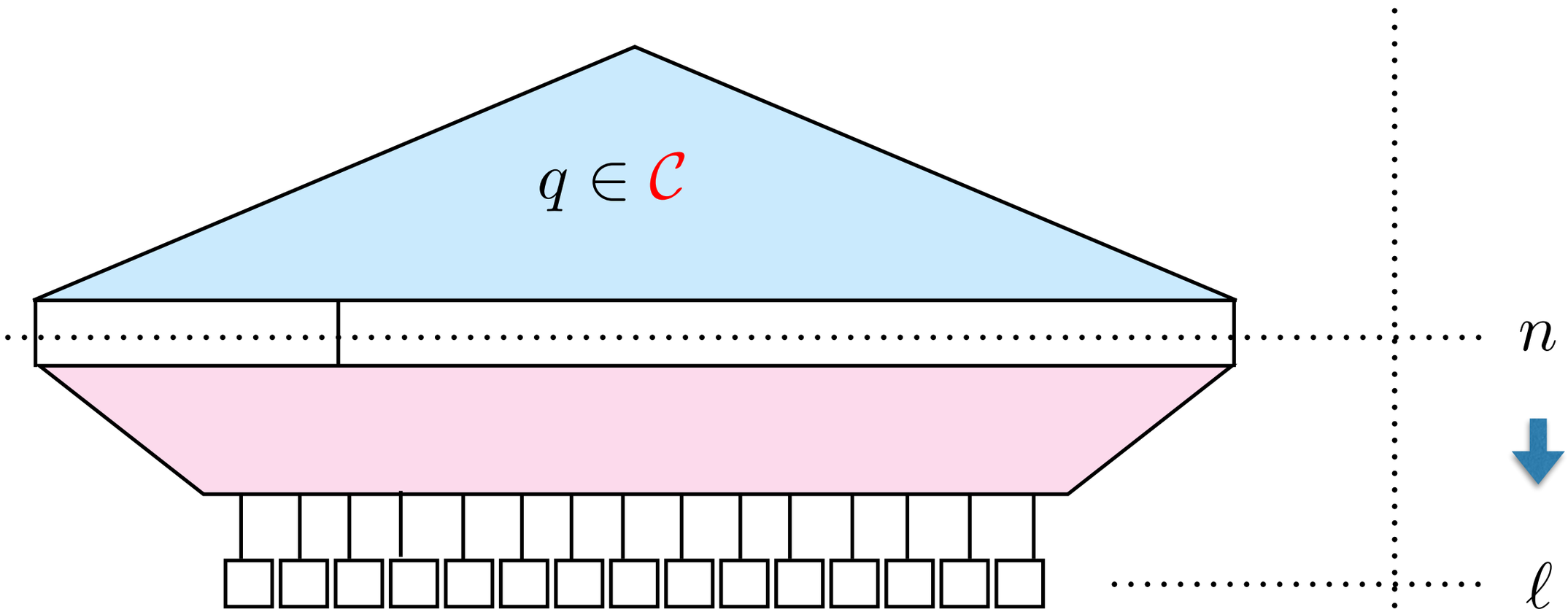

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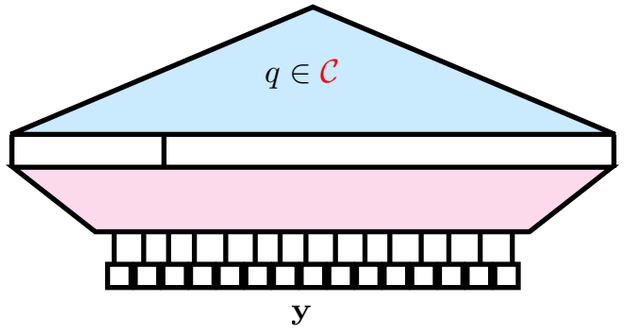

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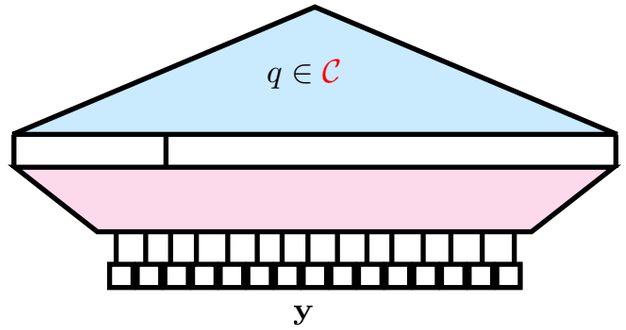

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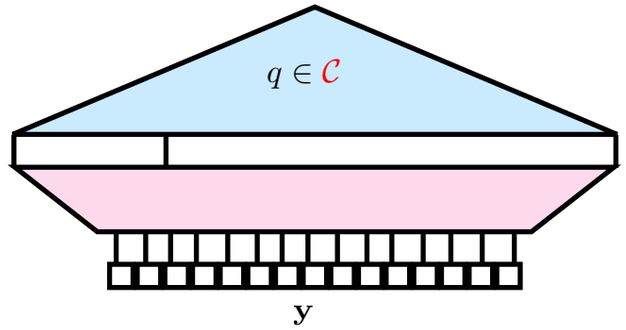

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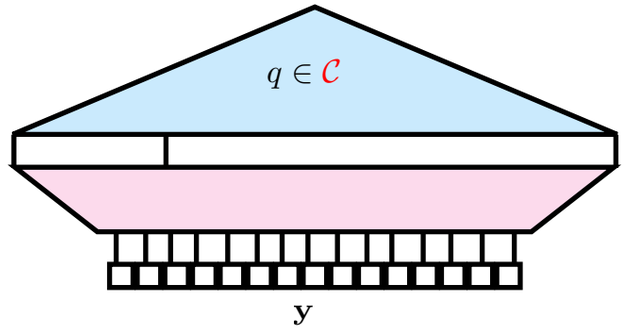

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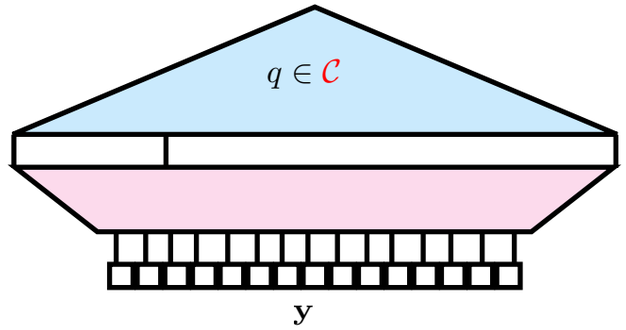

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Schwartz-Zippel

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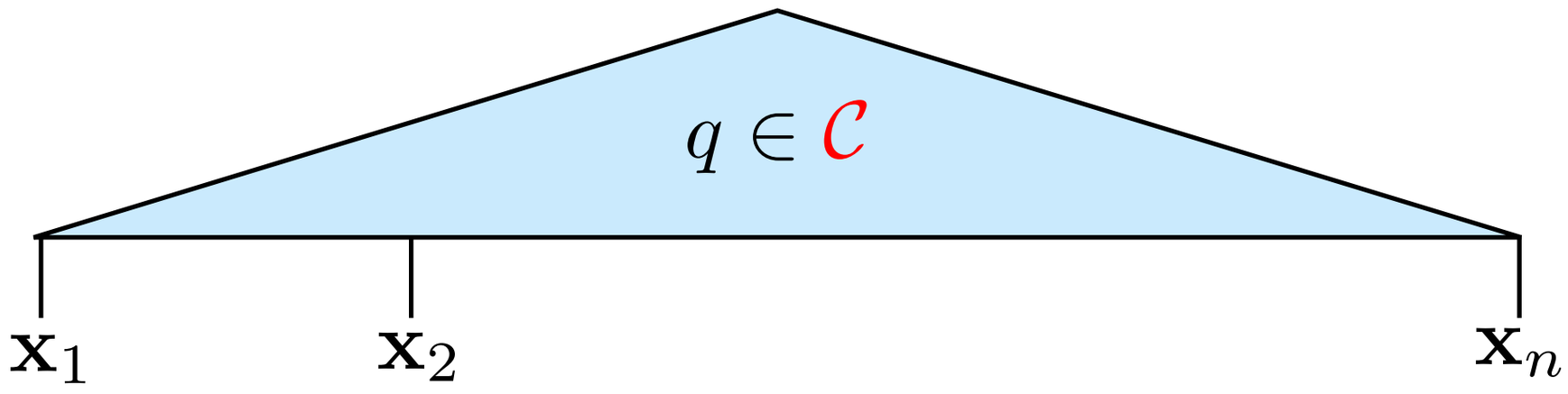

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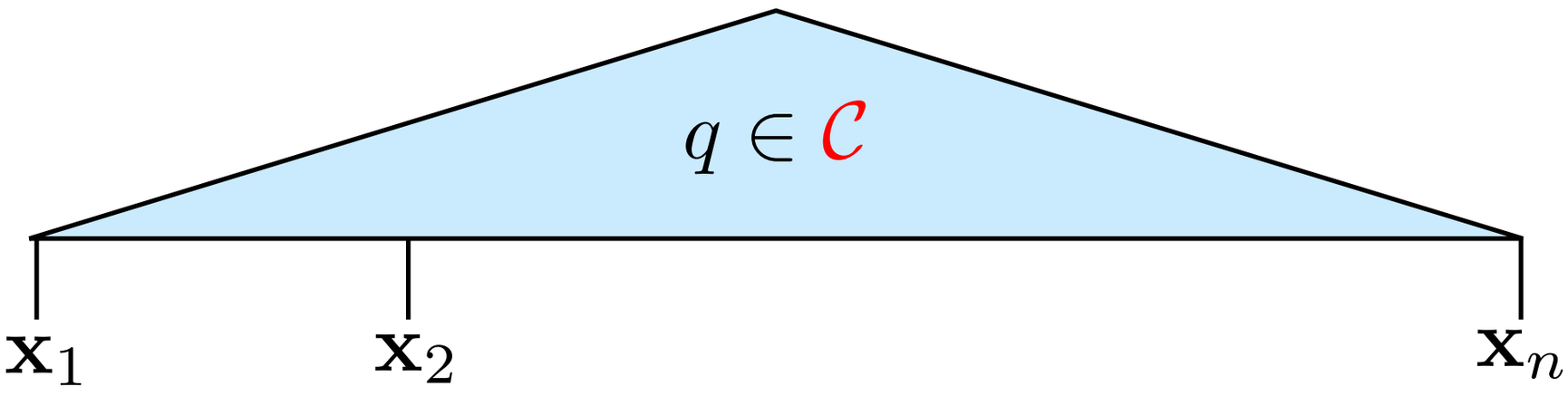
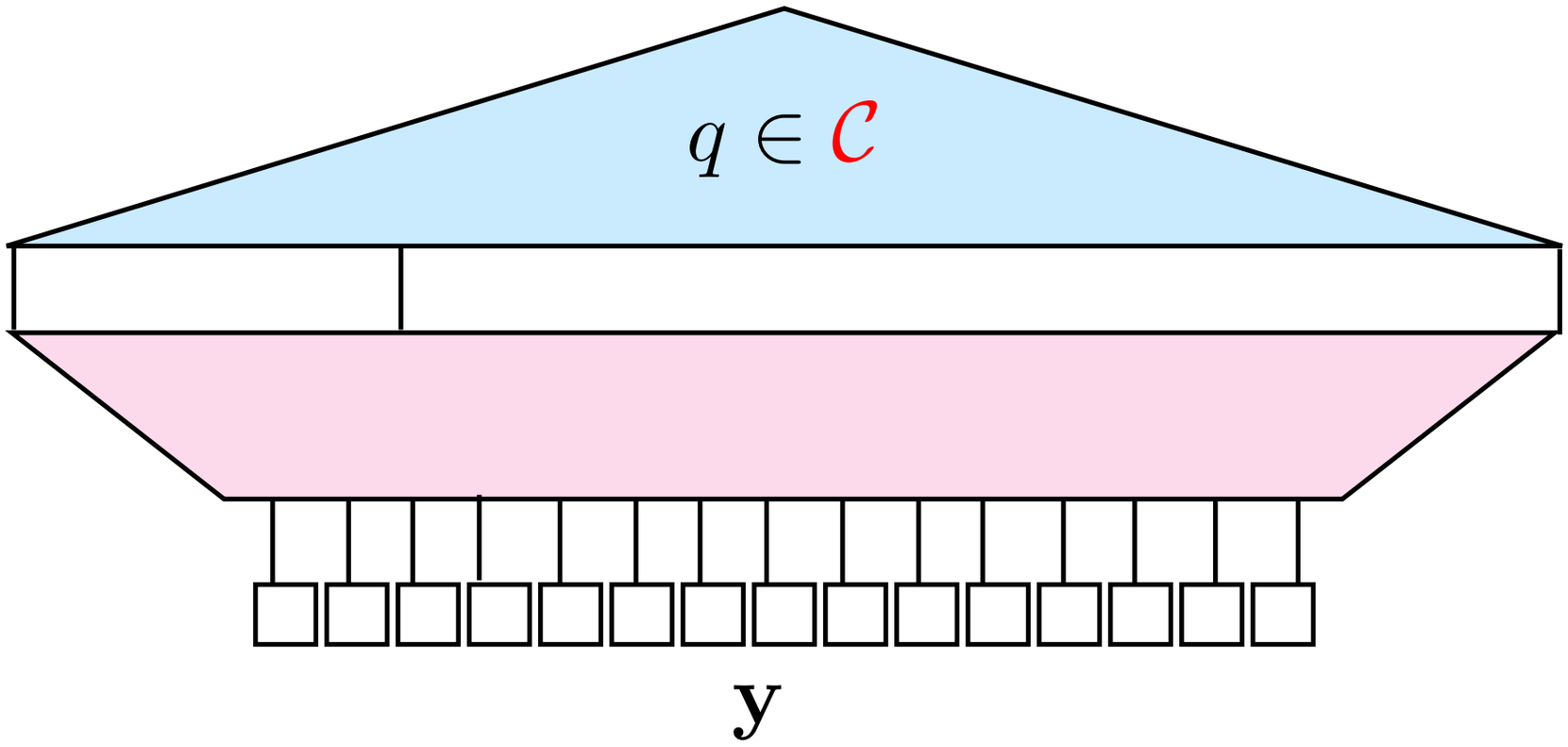
If $Q(\mathbf{y}) = q\big(f(\mathbf{y}|S_1), f(\mathbf{y}|S_2), \ldots, f(\mathbf{y}|S_n)\big) \equiv 0$



$\not\equiv 0$



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Proof sketch of the key lemma

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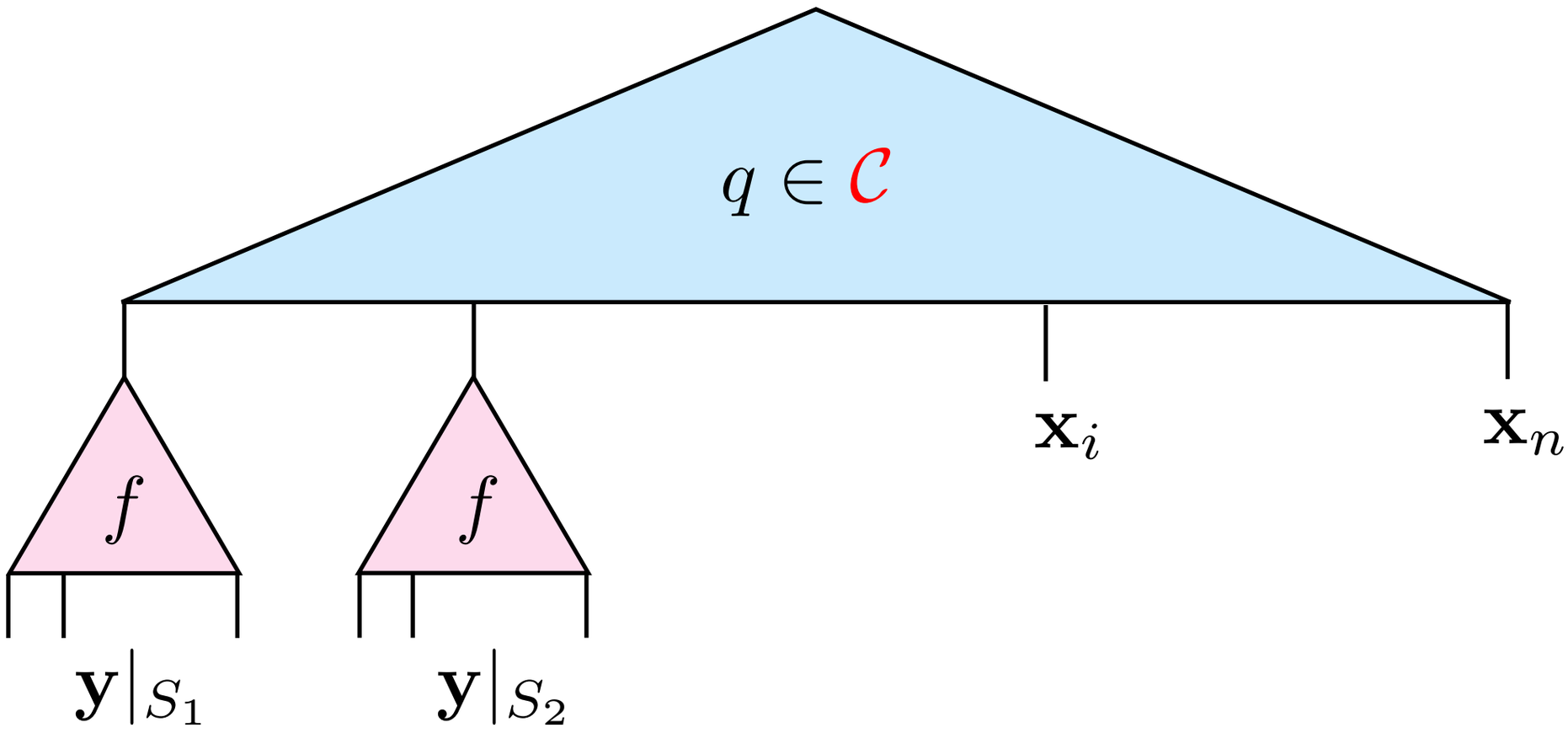

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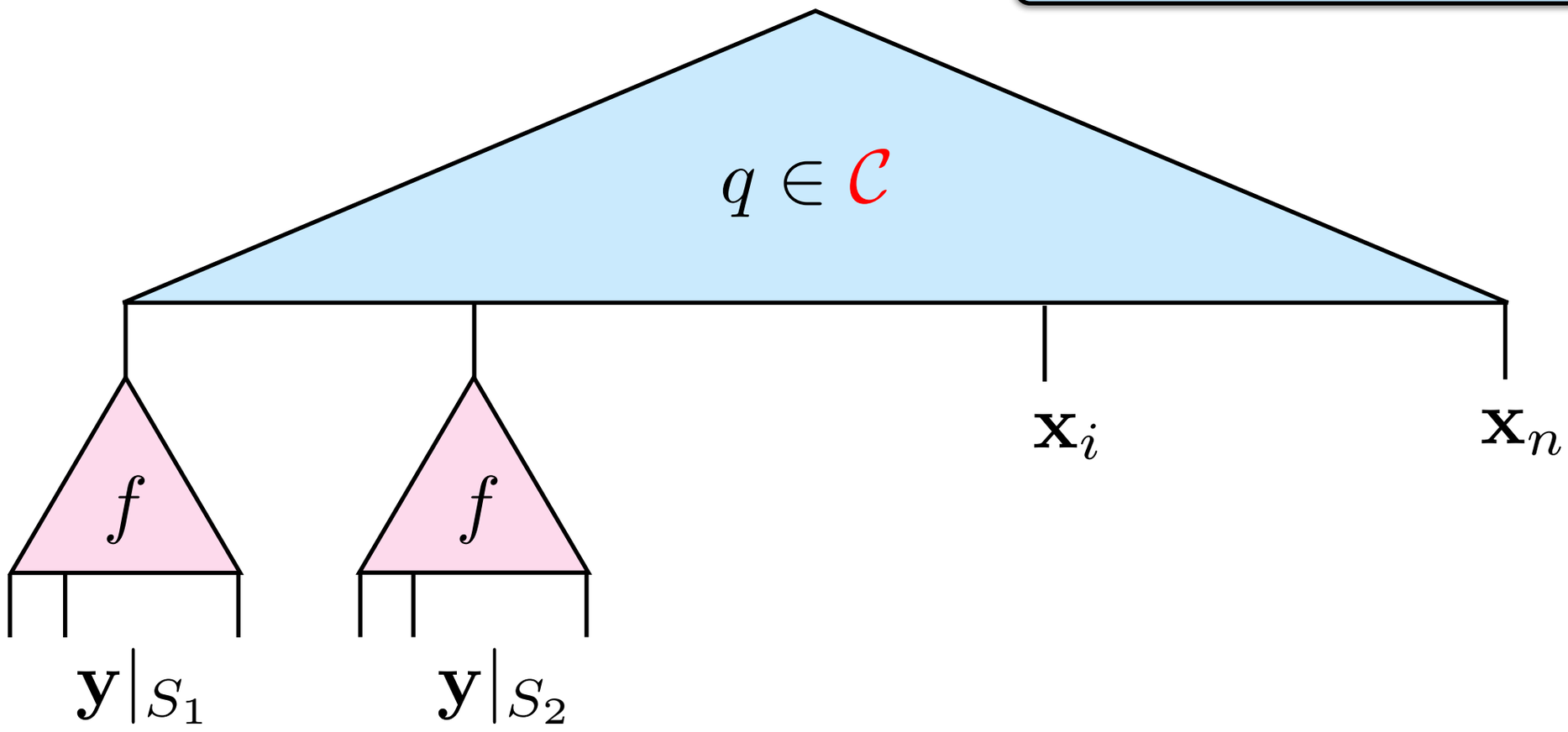
$$\mathbf{z} = \{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_n, \mathbf{y}\}$$



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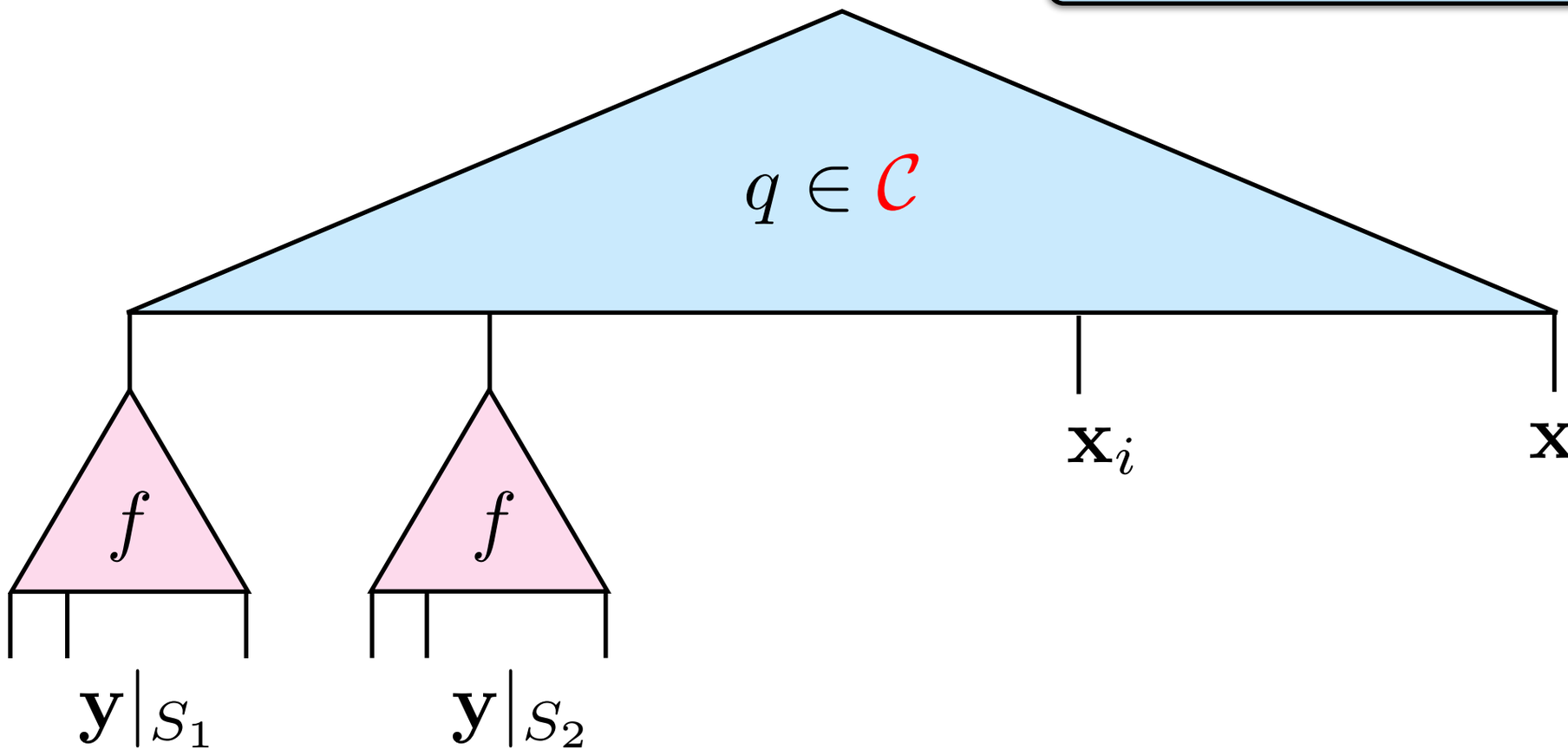

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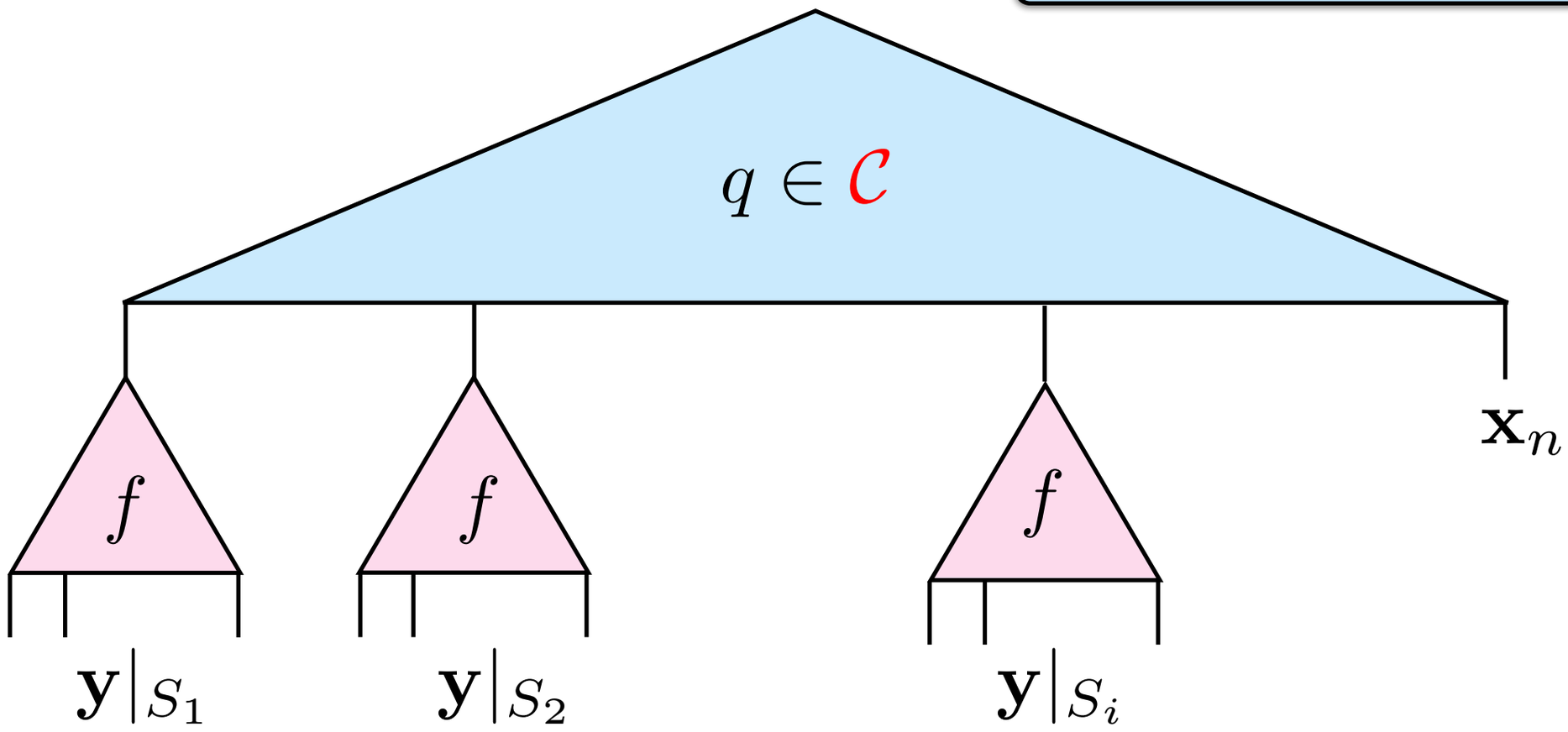

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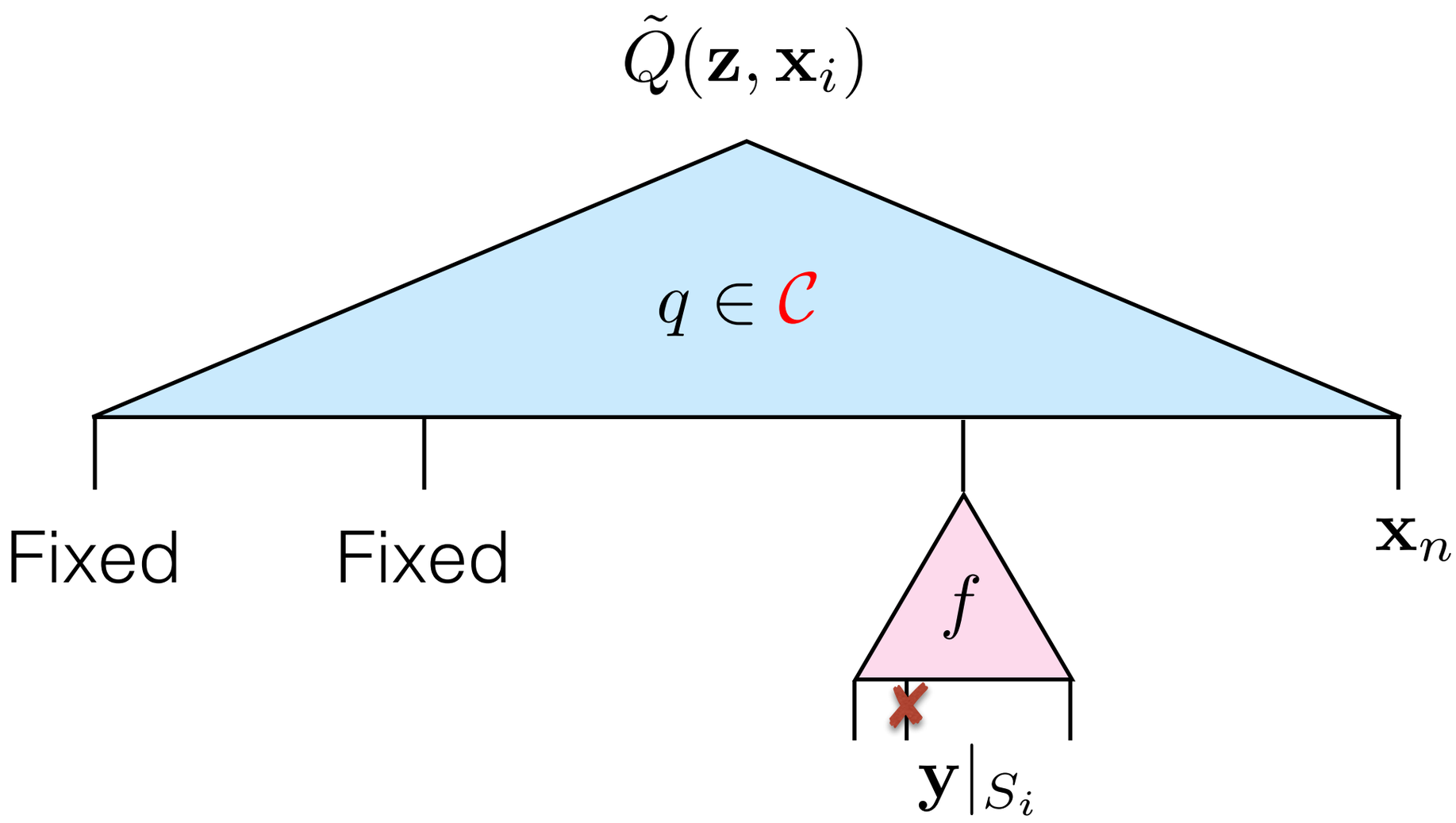

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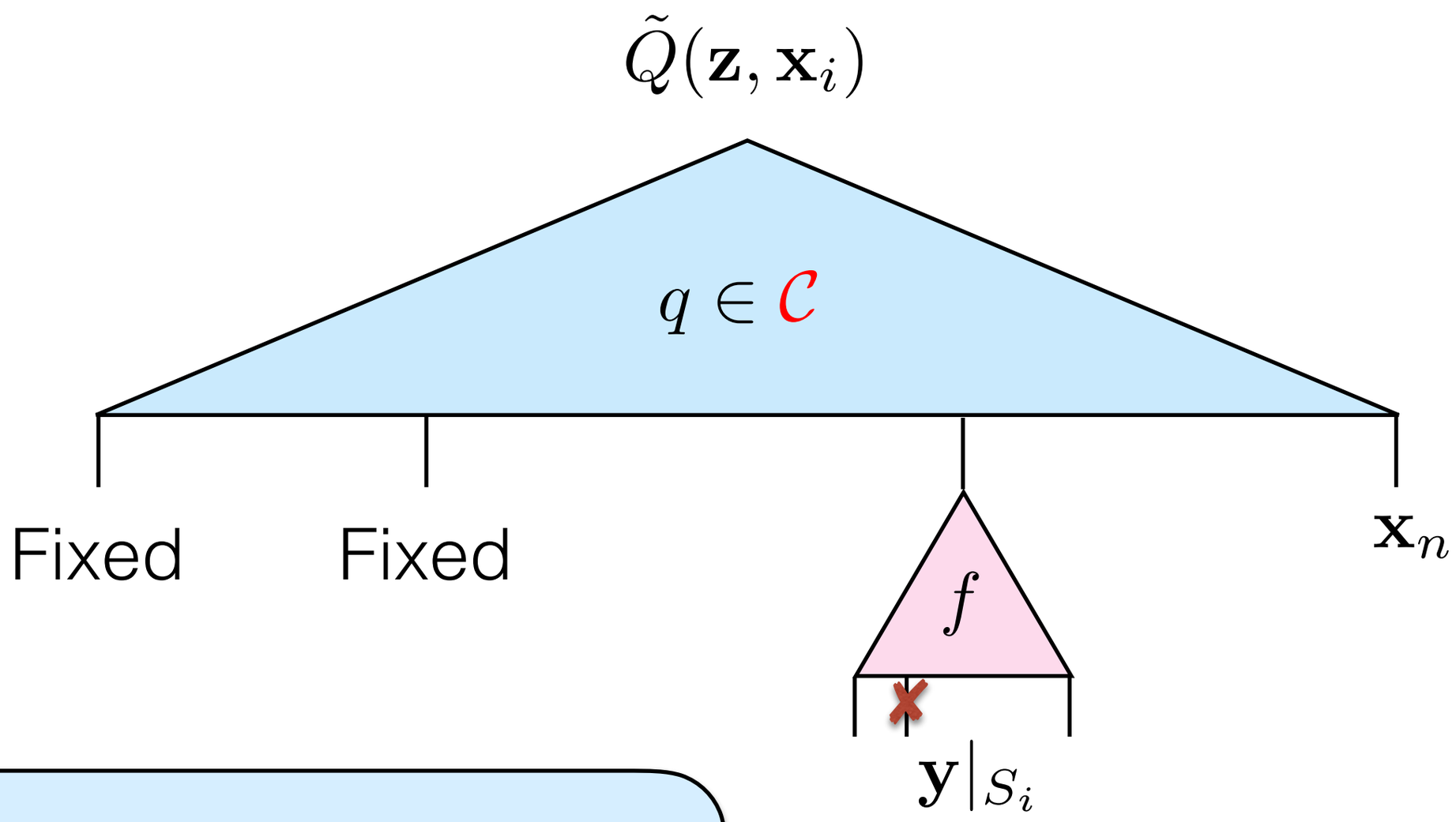

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Reducing to **polynomial factorization!**

Outline

- Arithmetic circuits and algebraic complexity classes
- Polynomial identity testing (PIT)
- Hardness vs Randomness for arithmetic circuits
- Polynomial factorization
- Open problems

Polynomial factorization (Simplified setting)

Goal: For any $P(\mathbf{z}, y) \in \mathcal{C}$ such that $P(\mathbf{z}, f(\mathbf{z})) = 0$.
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non-deterministic (existential)

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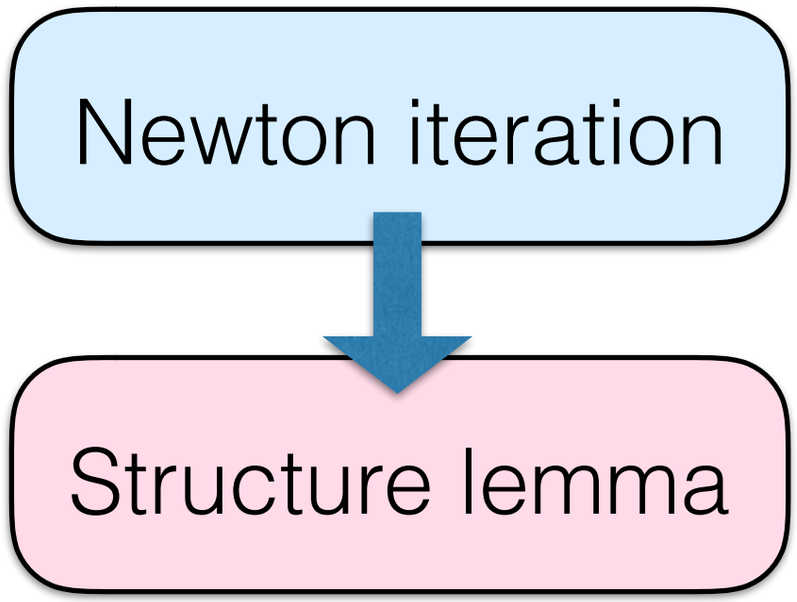

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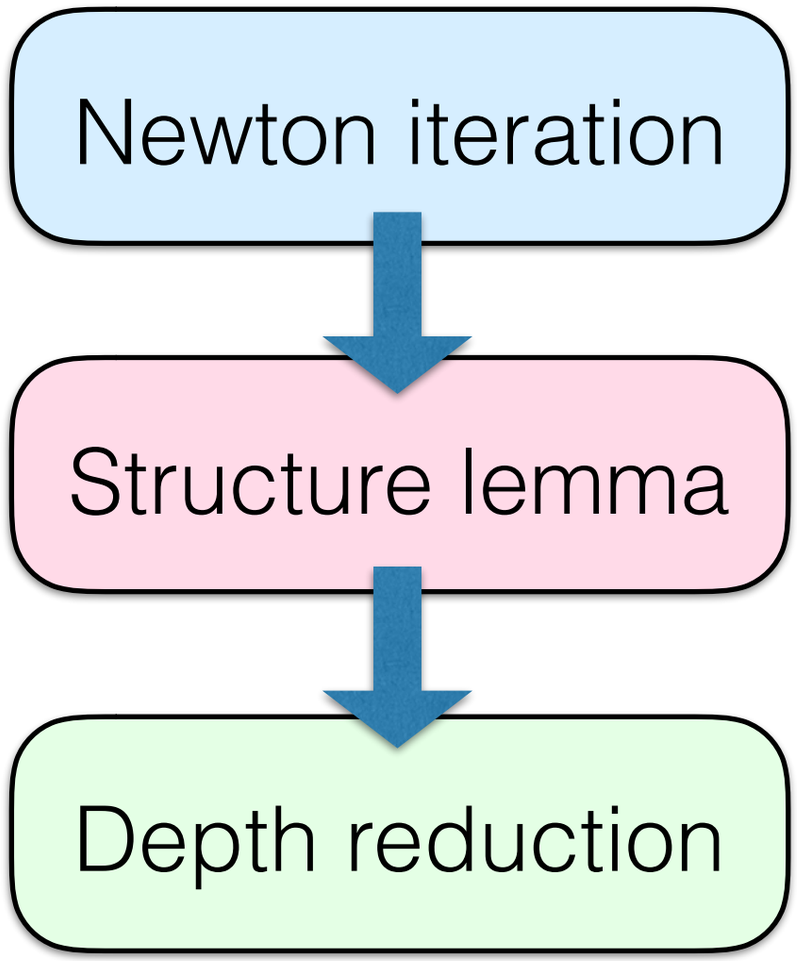

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Q: How to efficiently update?

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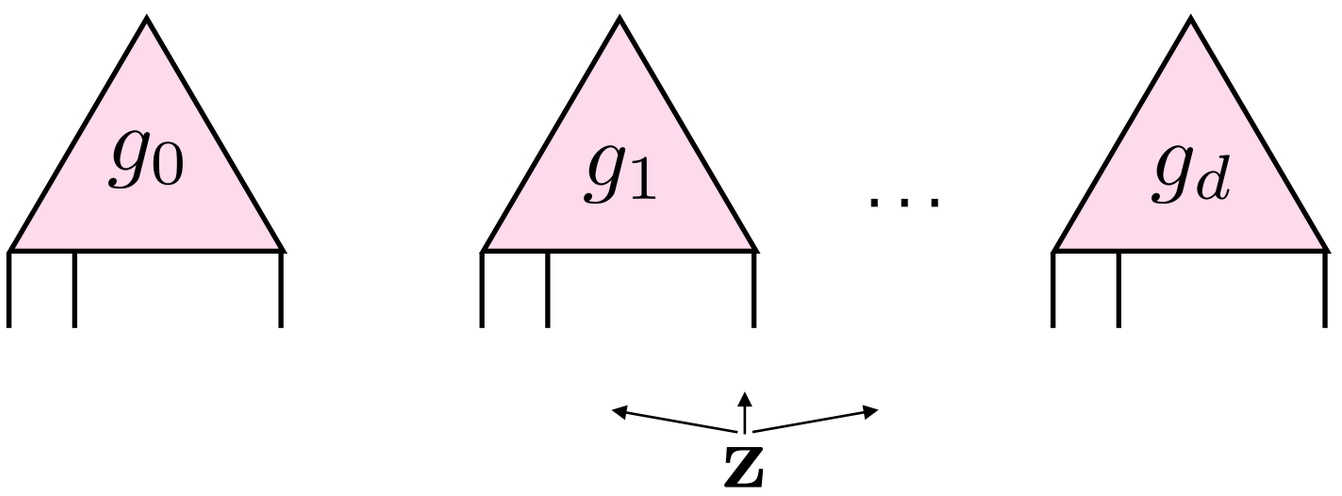

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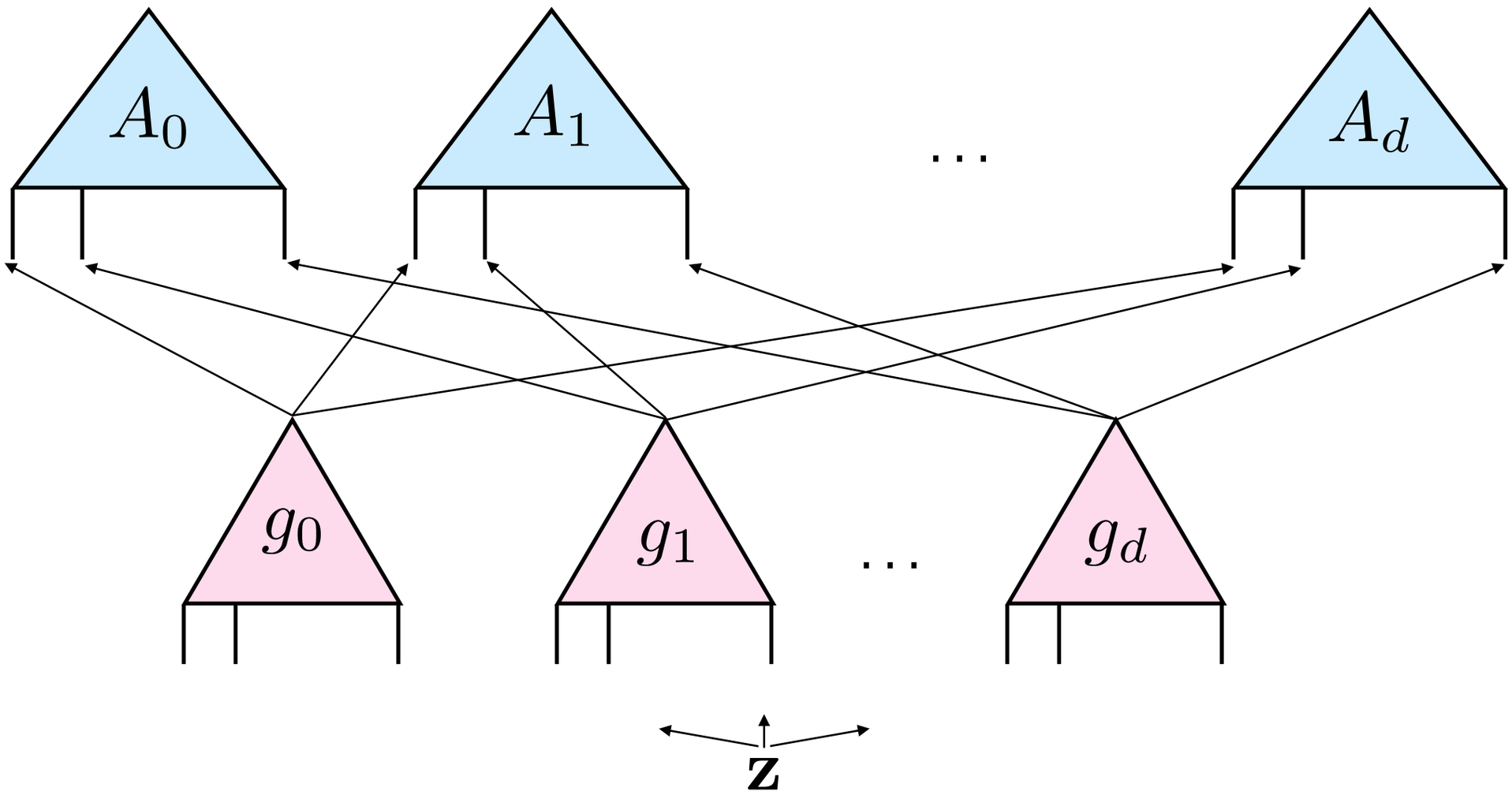

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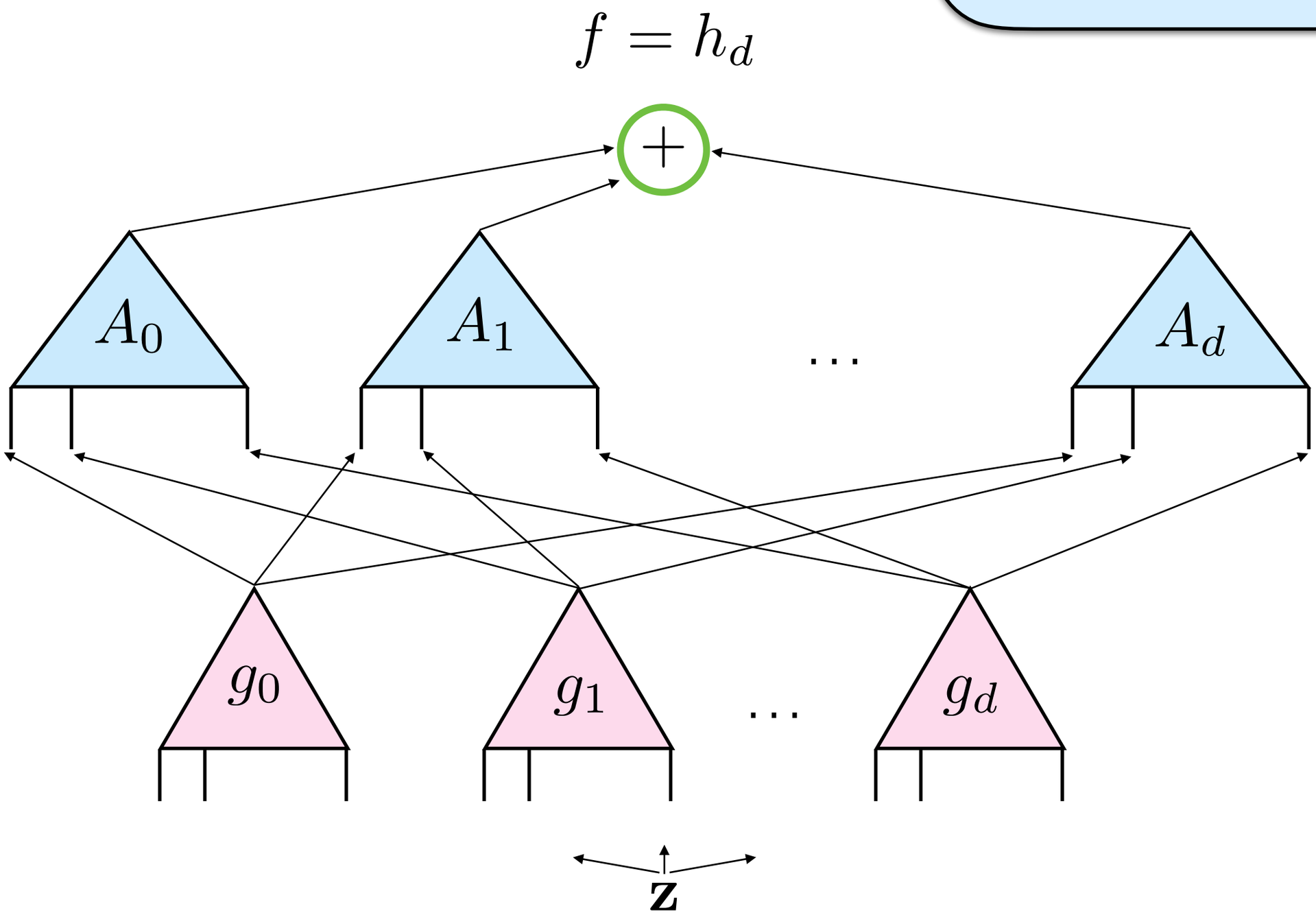

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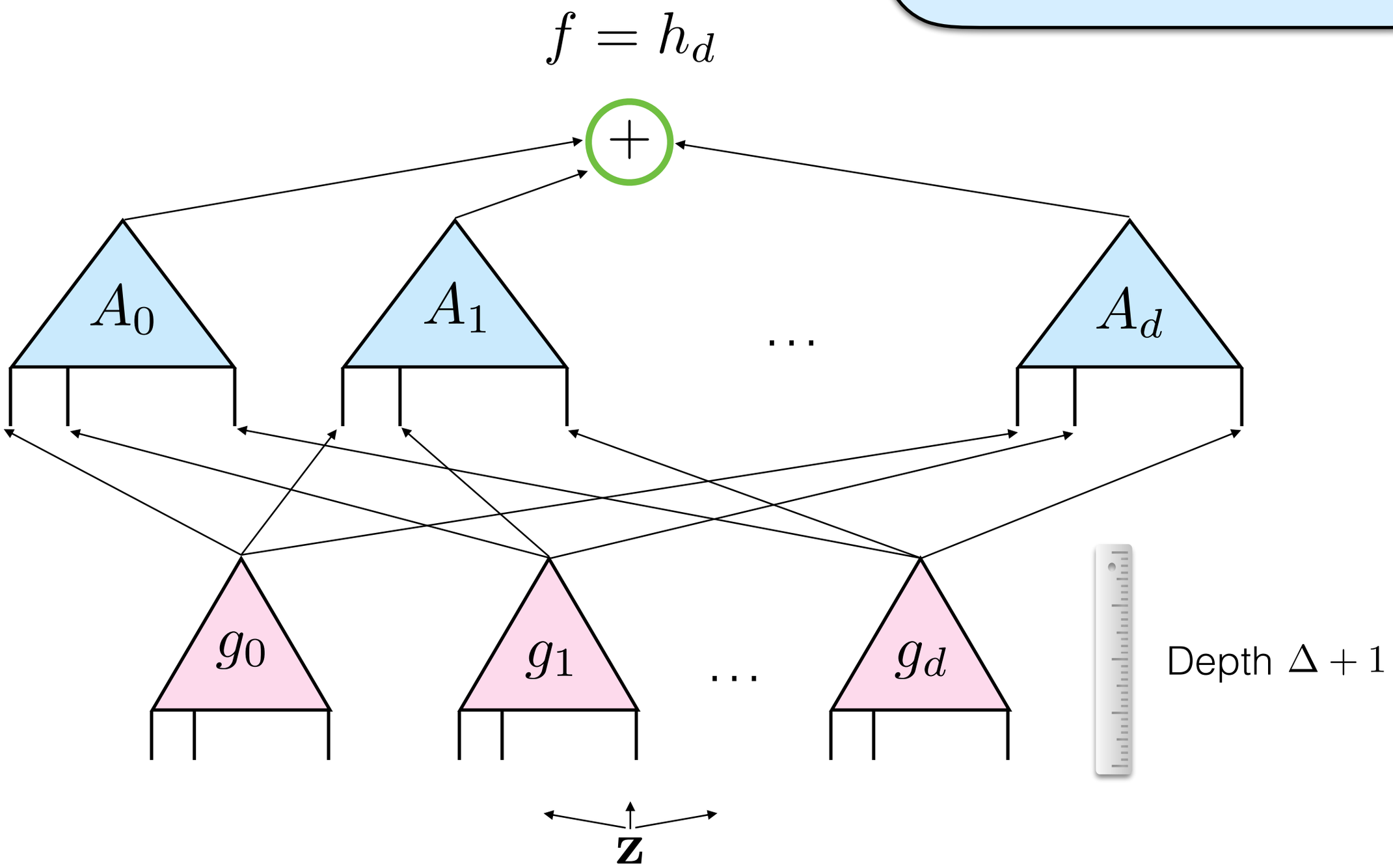

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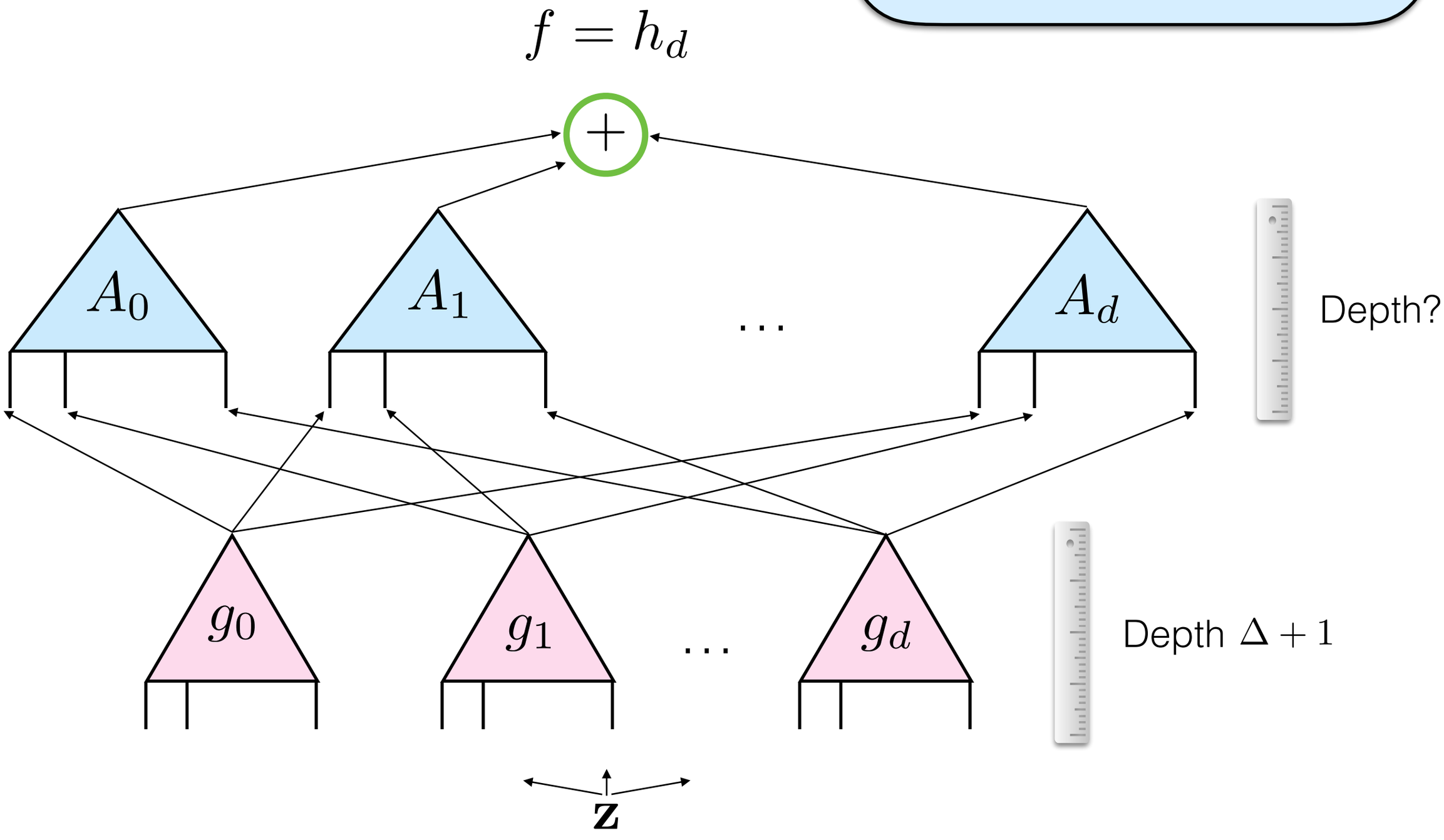

Depth reduction [Gupta-Kamath-Kayal-Saptharishi'13]

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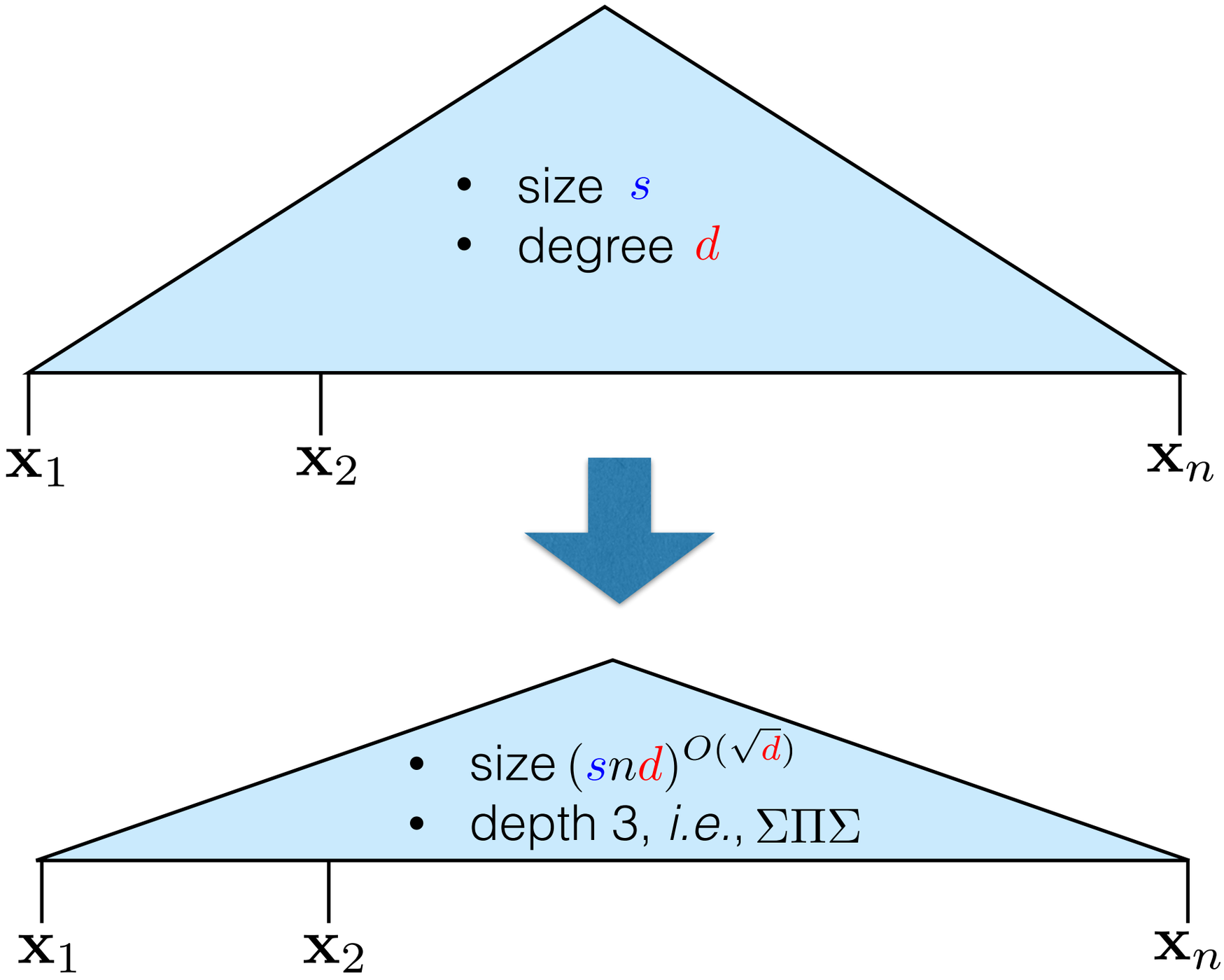
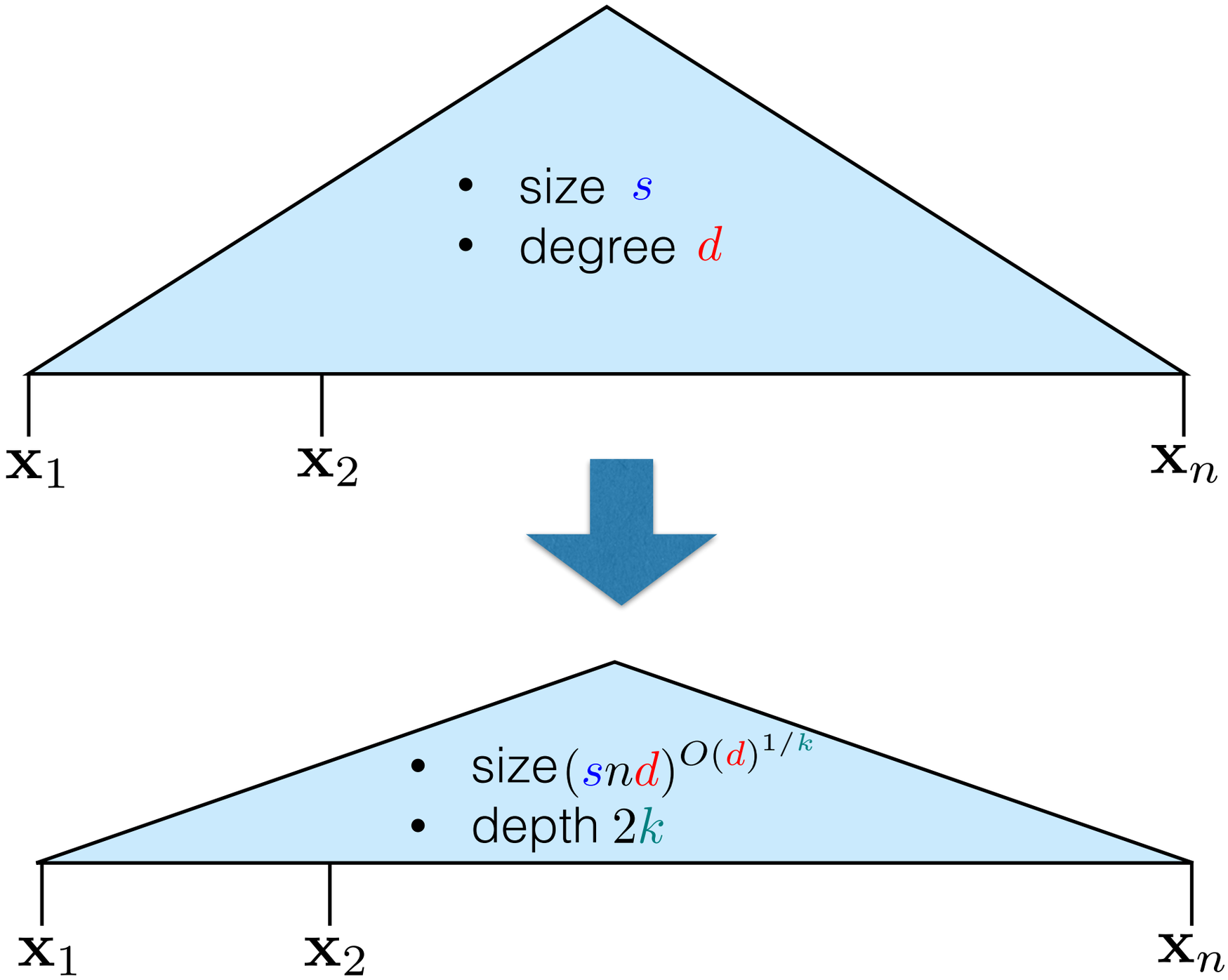
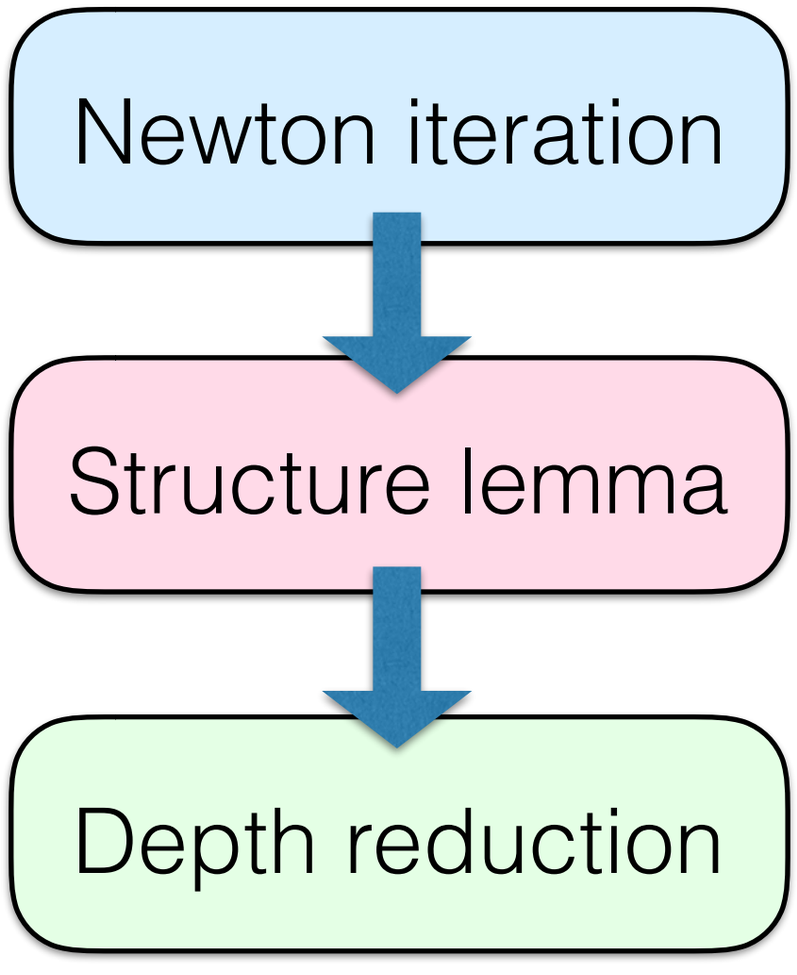


Depth reduction
[Agrawal-Vinay'08, Koiran'12, Tavenas'13]



Factorization for bounded depth circuits (Wrap up)

Goal: For any $P(\mathbf{z}, y) \in \text{Depth-}\Delta$ s.t. $P(\mathbf{z}, f(\mathbf{z})) = 0$.
Show that $f \in \text{Depth-}\Delta + O(1)$.



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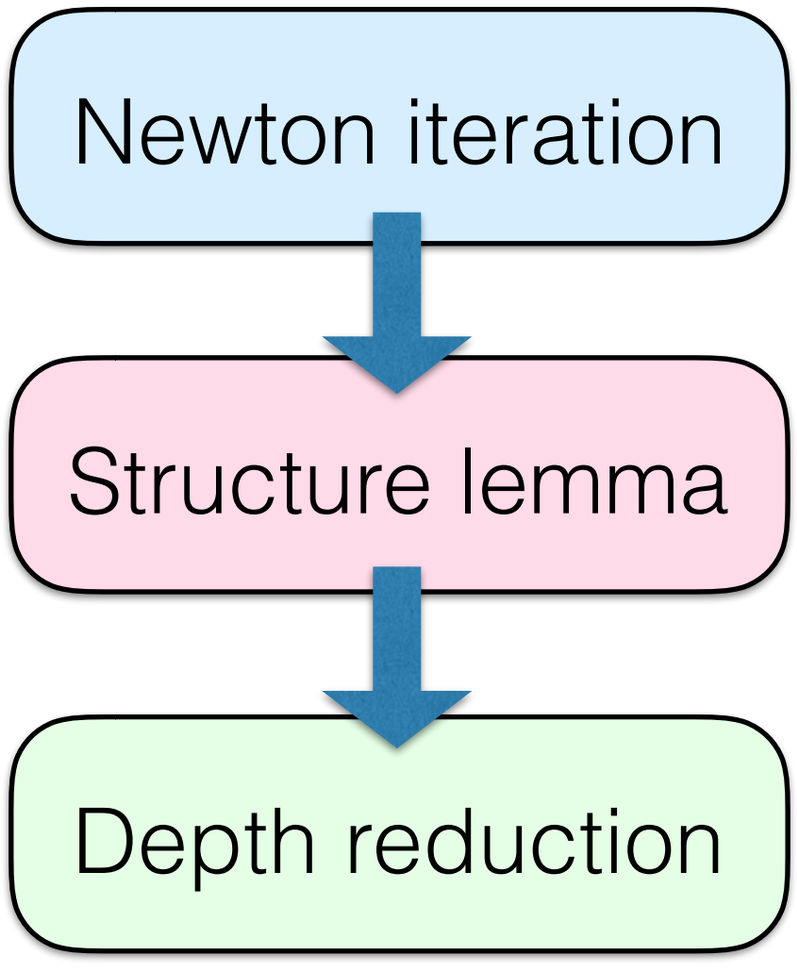

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	[DSY'09]	This work
Lower bound for $\text{Depth-}\Delta$		With degree $O(\log^2 n / \log^2 \log n)$
PIT for $\text{Depth-}\Delta - 5$	With bounded individual degree	

Outline

- Arithmetic circuits and algebraic complexity classes
- Polynomial identity testing (PIT)
- Hardness vs Randomness for arithmetic circuits
- Polynomial factorization
- Open problems

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Thank you!