# ODE-Inspired Analysis for the Biological Version of Oja's Rule in Solving Streaming PCA

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COLT 2020

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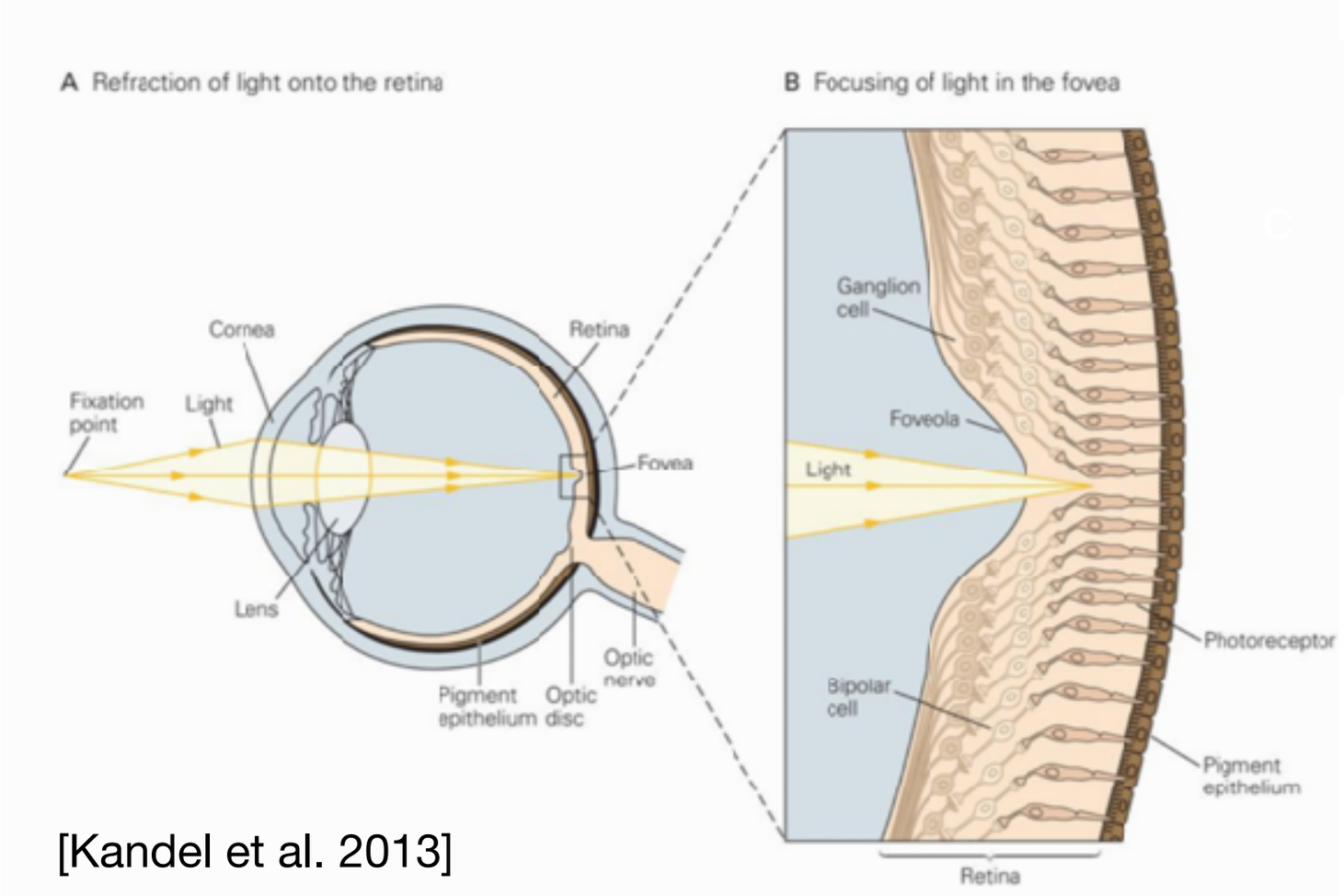
# Retina




[Kandel et al. 2013]

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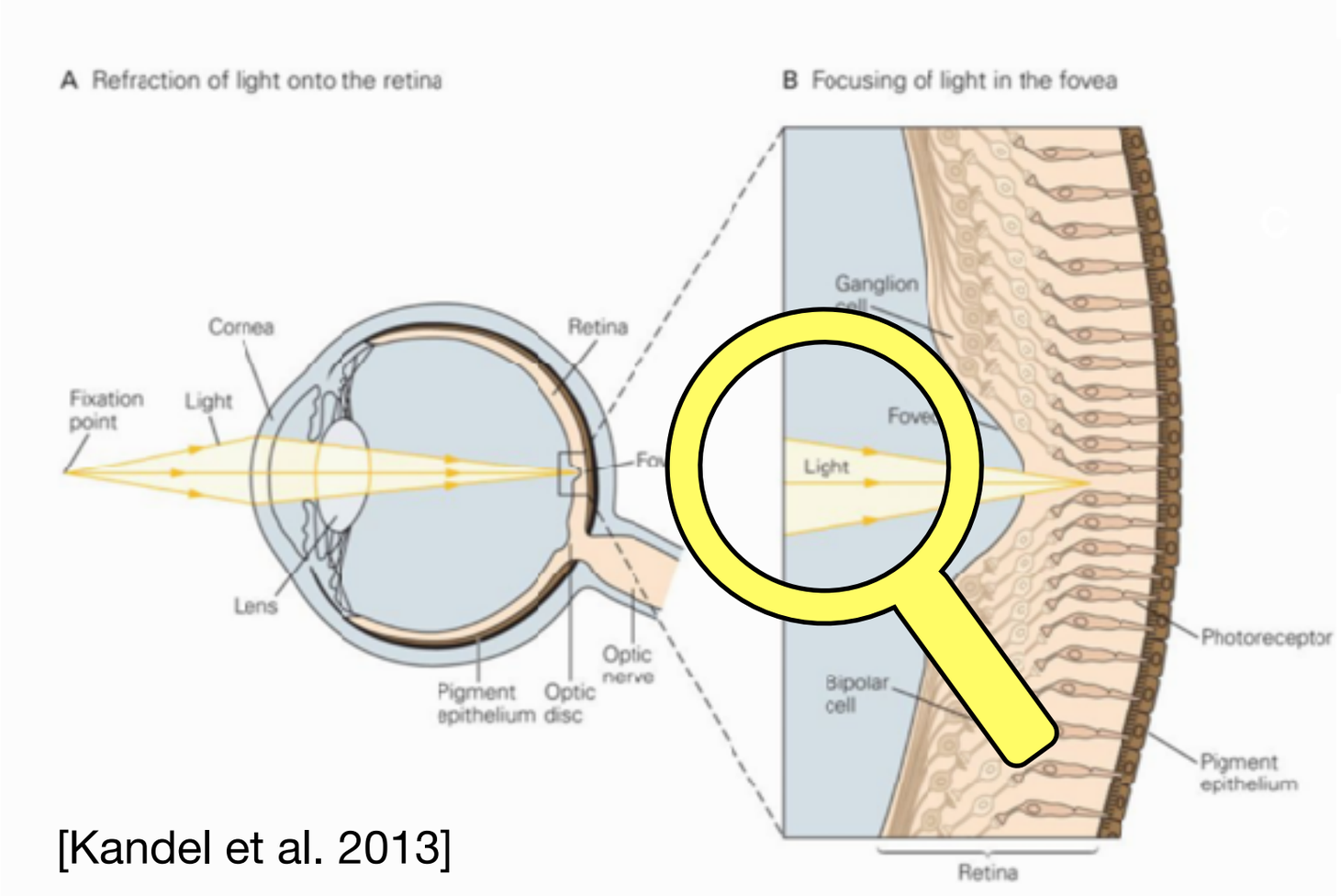


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**Light**

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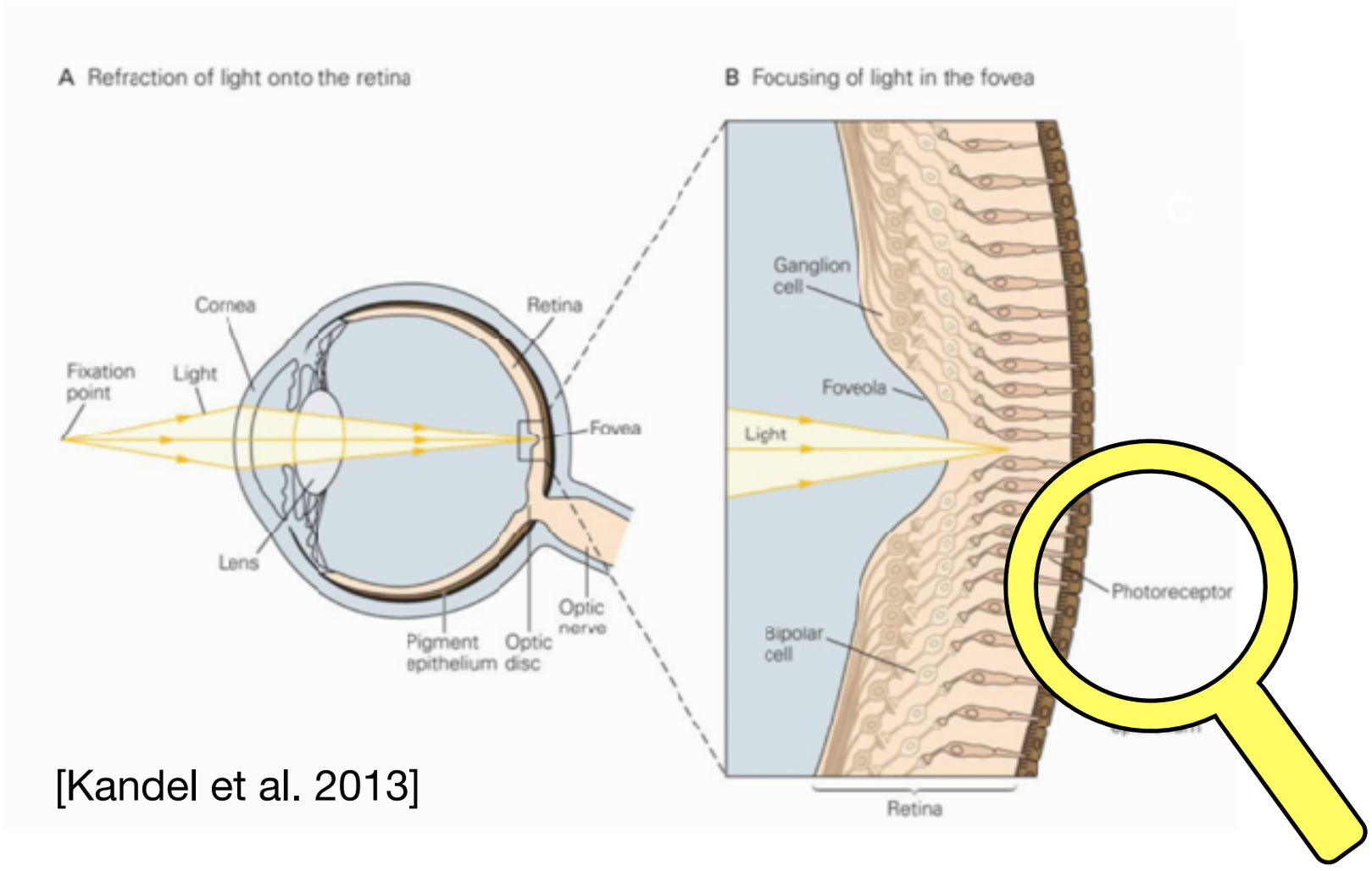


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**Light** → **Photoreceptors**

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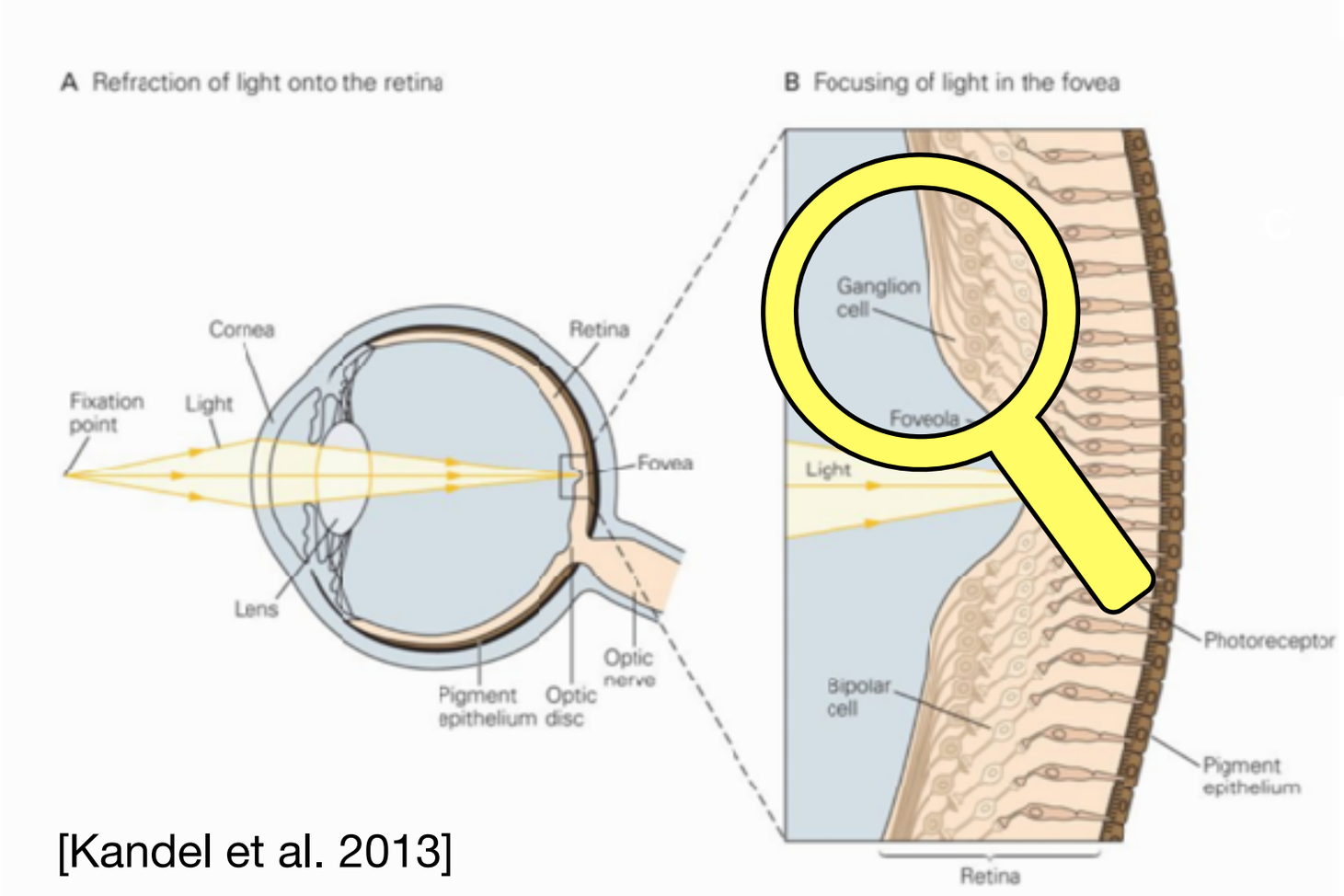


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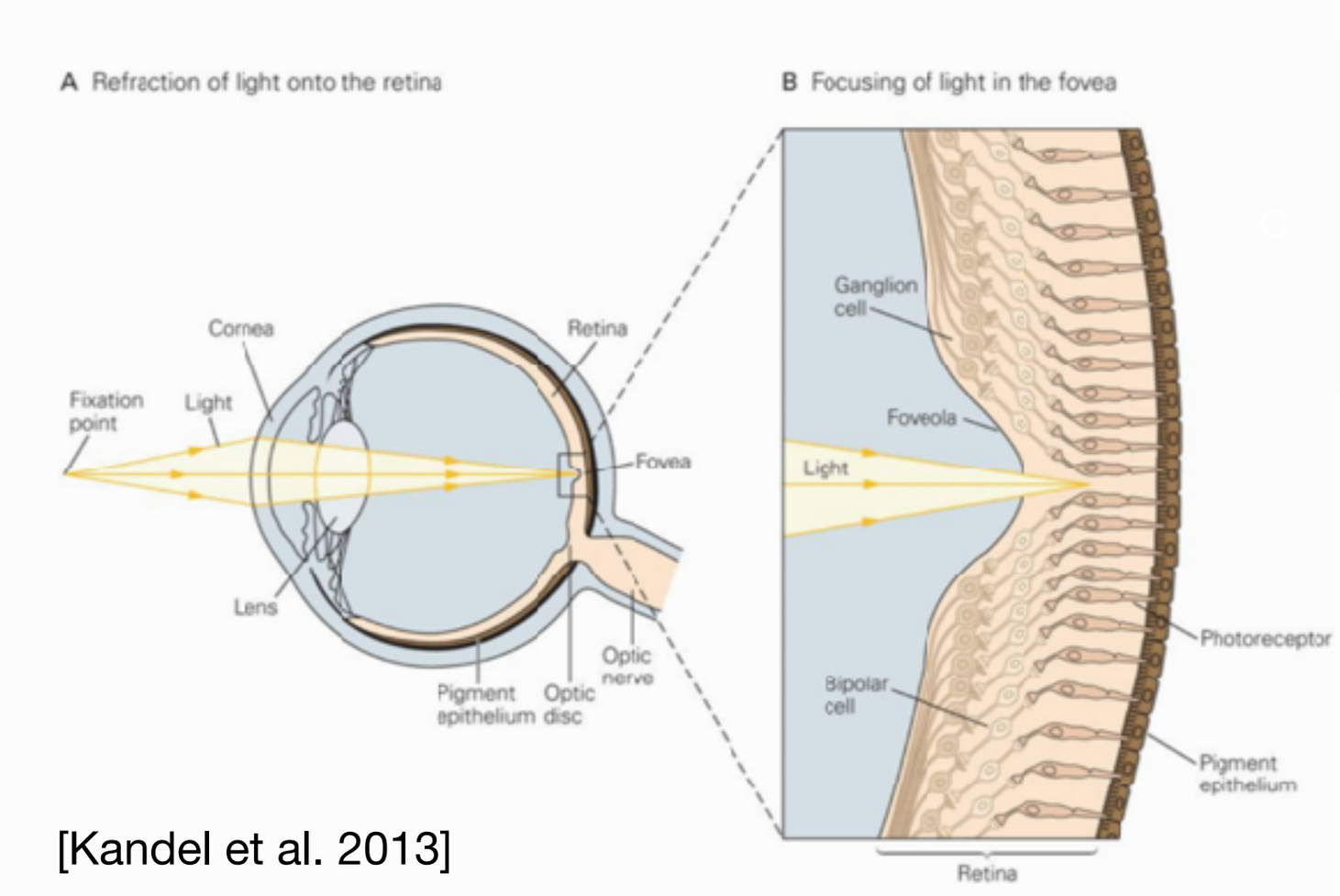


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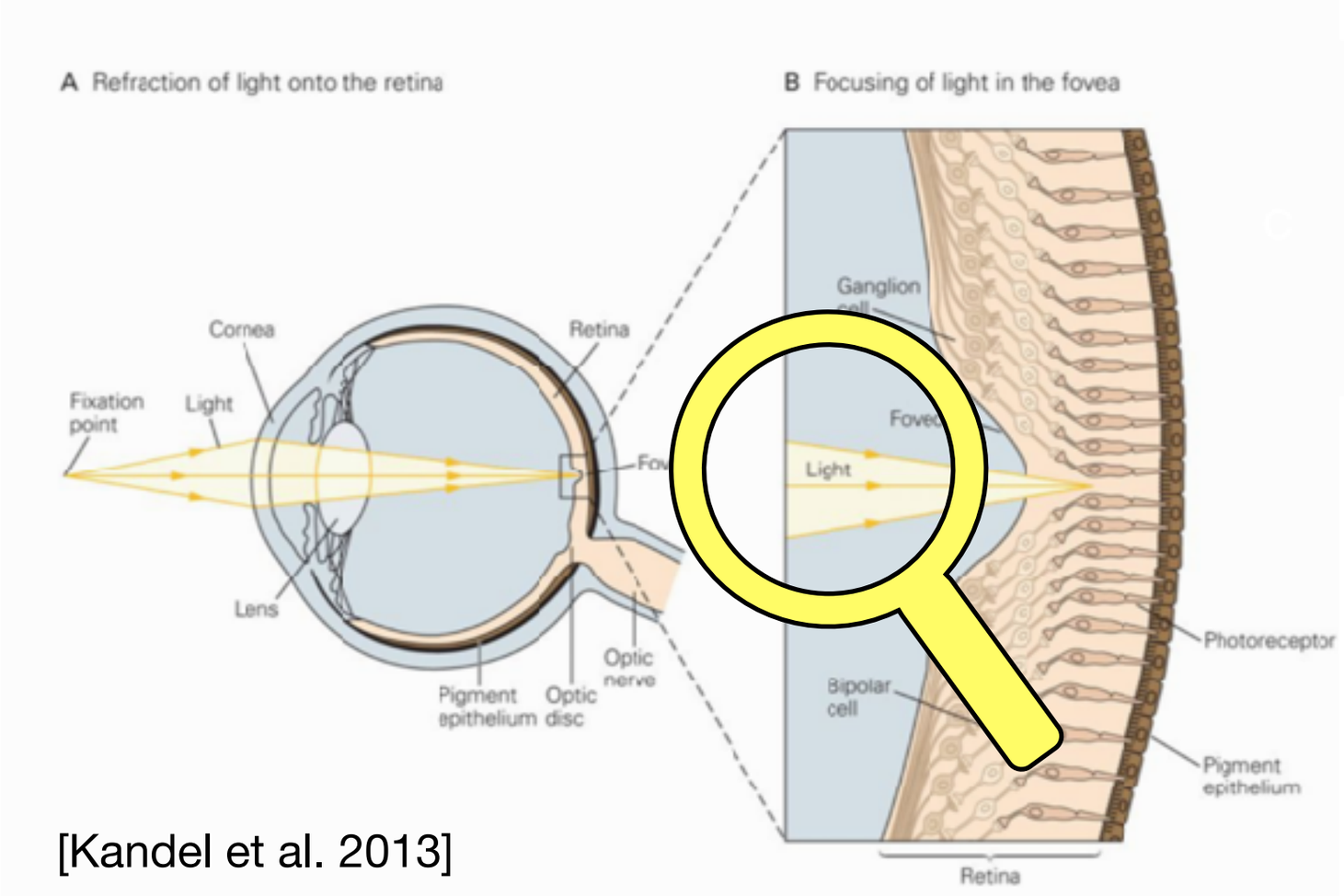


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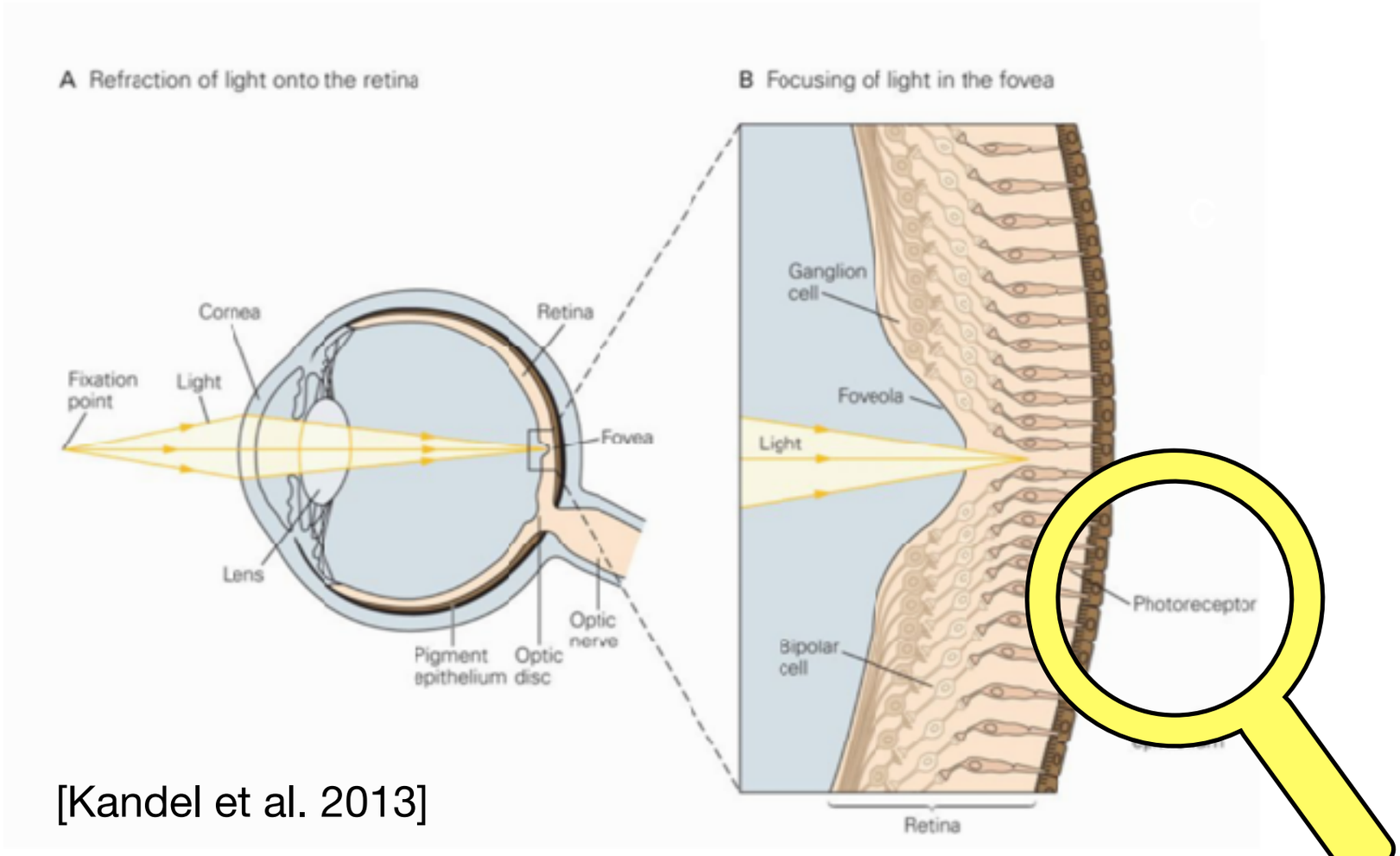


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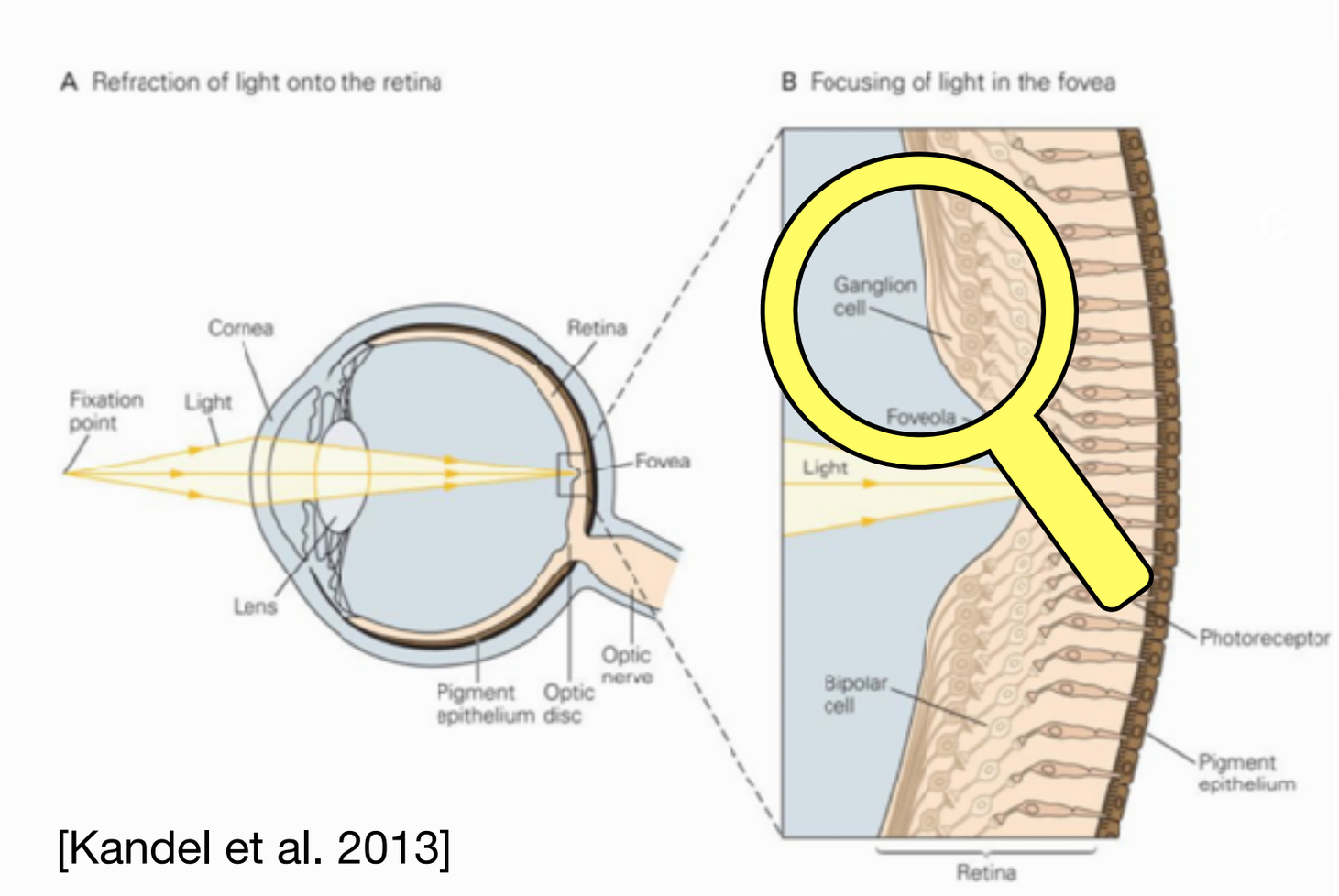


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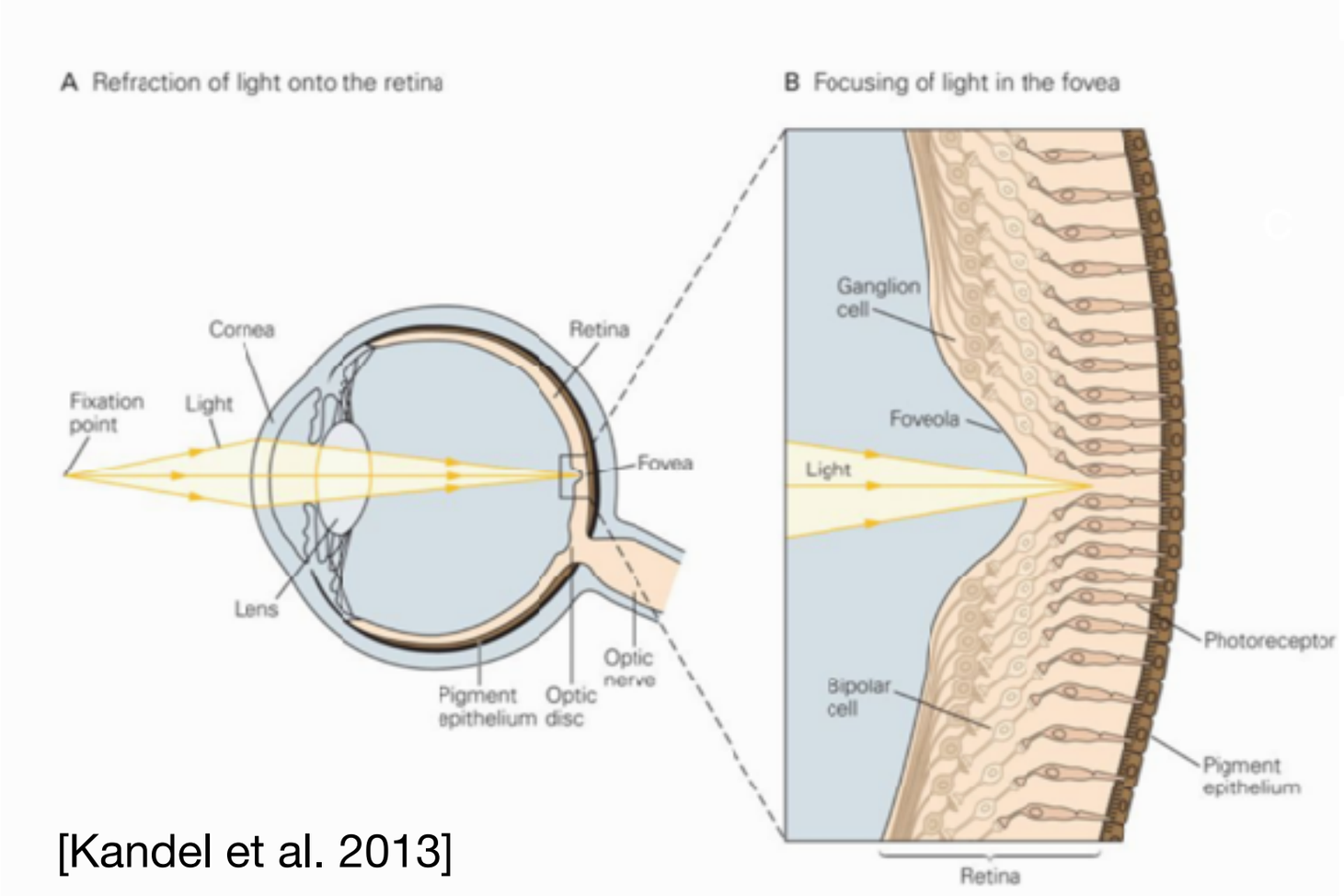


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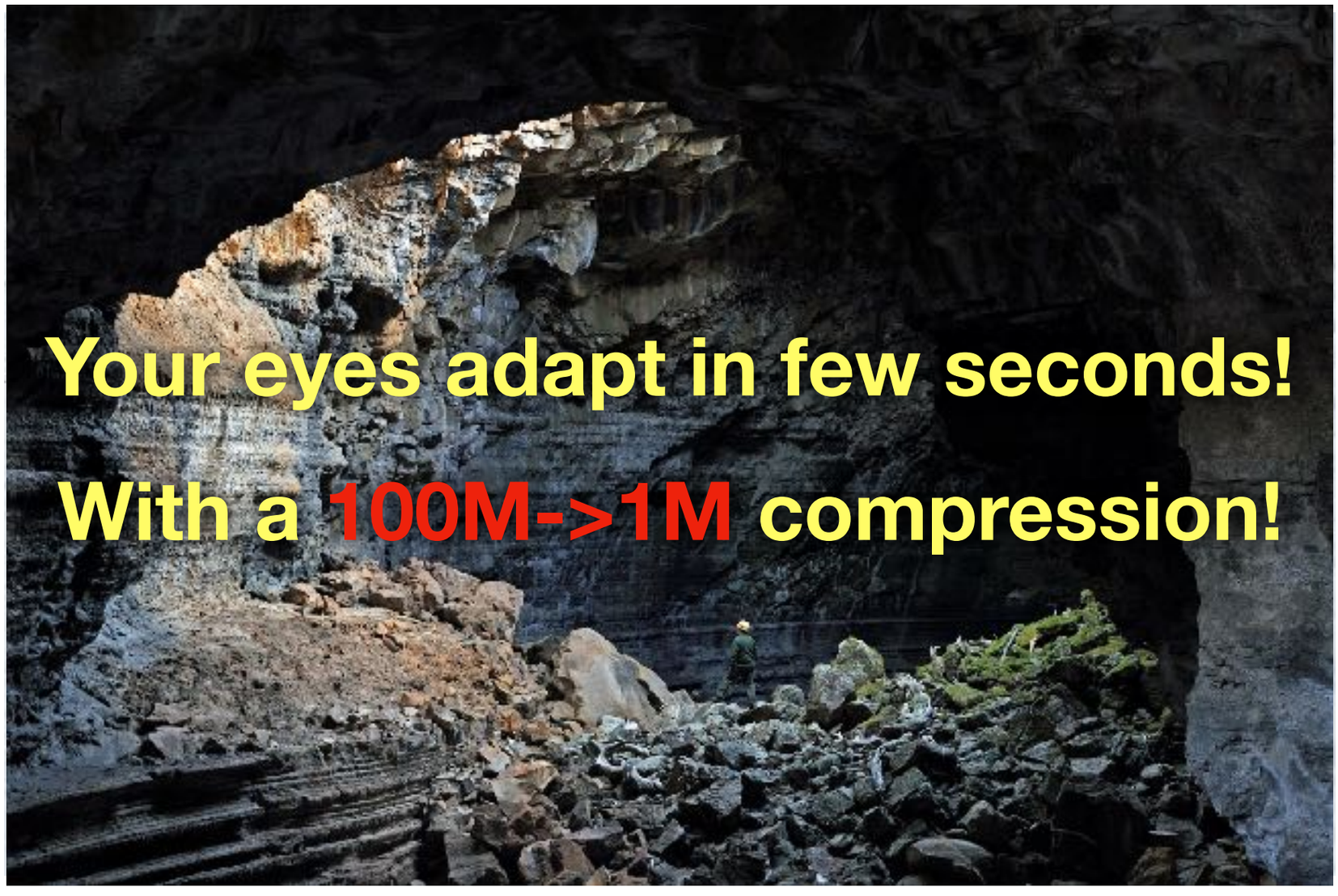

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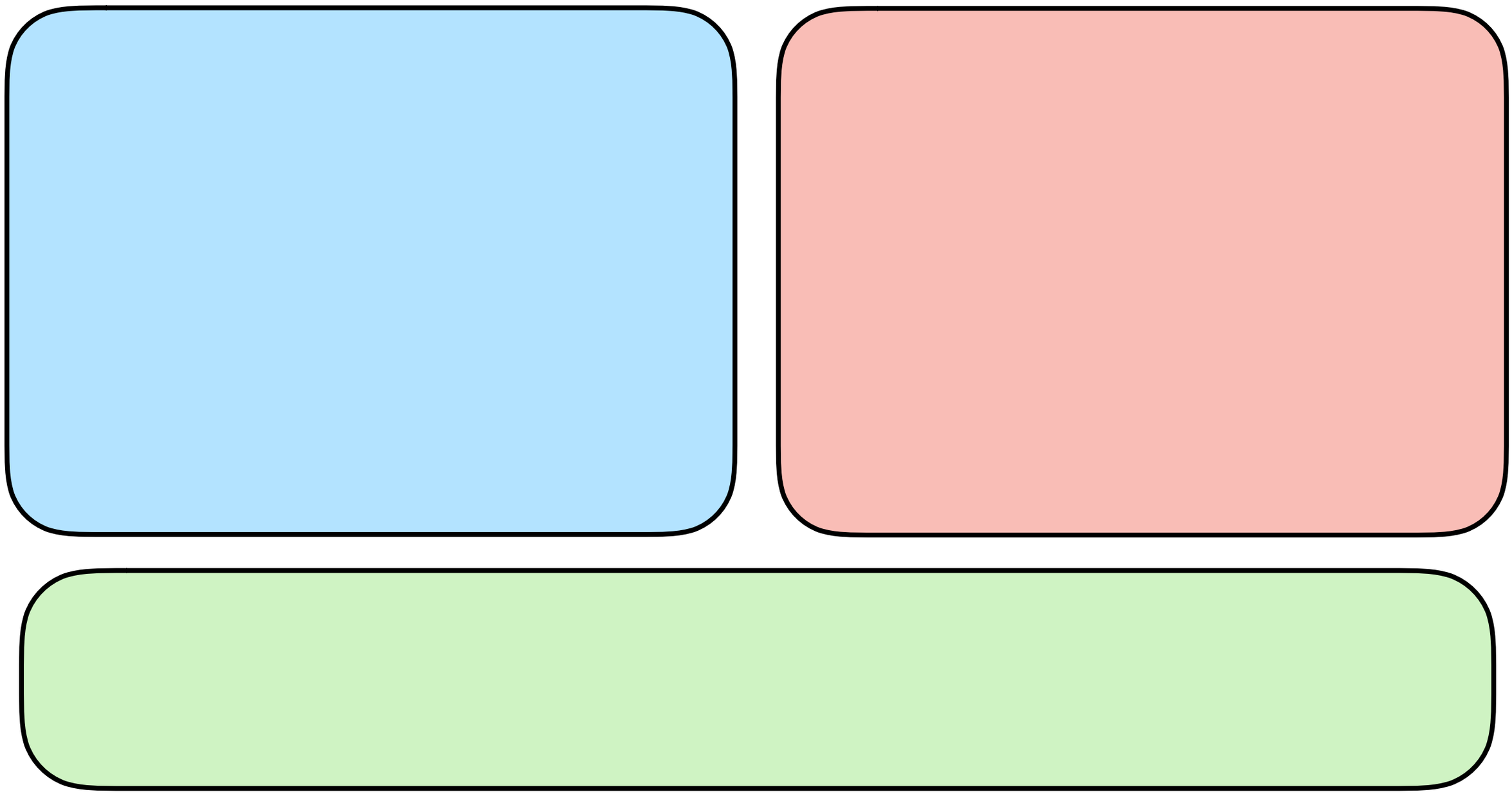

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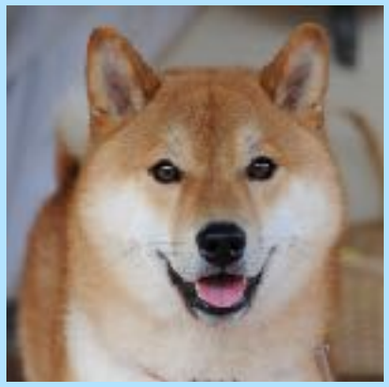

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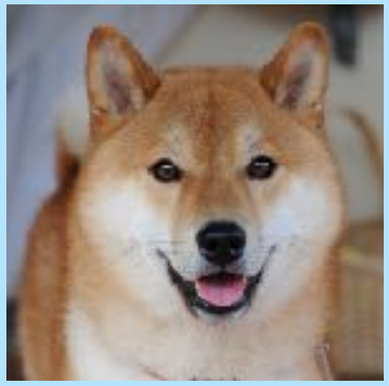

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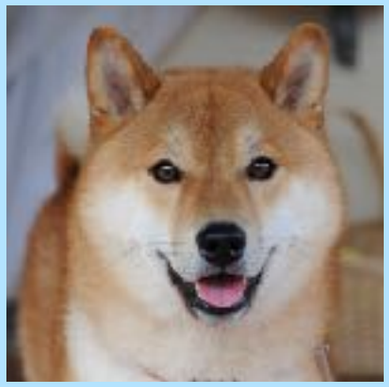

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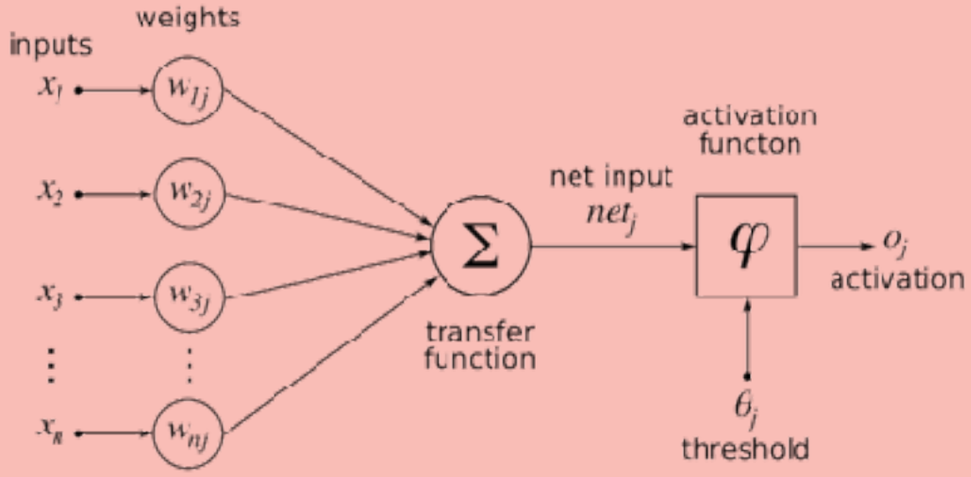

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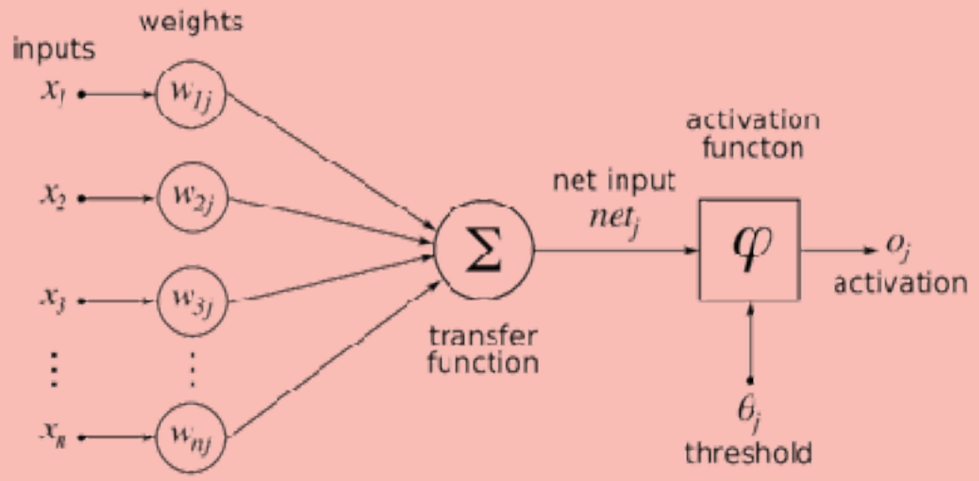


## Biologically-Realistic Timescale

(Adaptation happens in few seconds)

# Receptive Field

## (of a Ganglion Cell)

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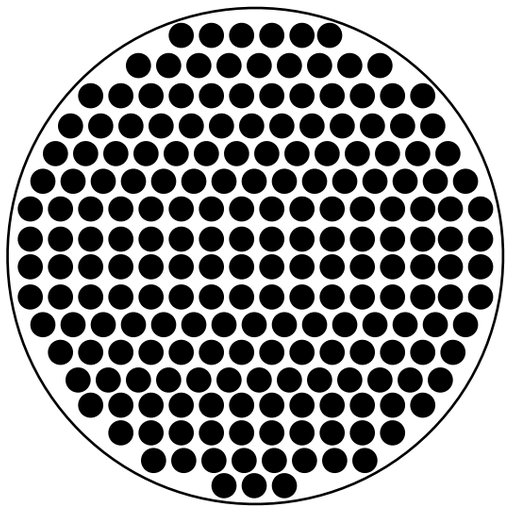

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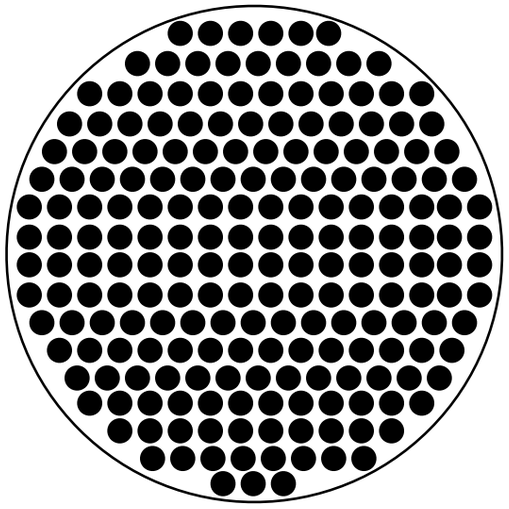

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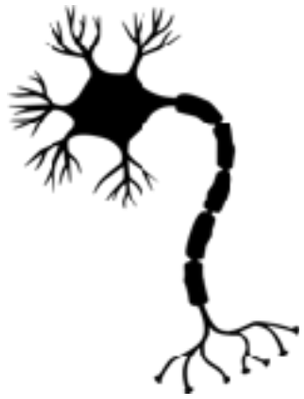

**Ganglion cell**

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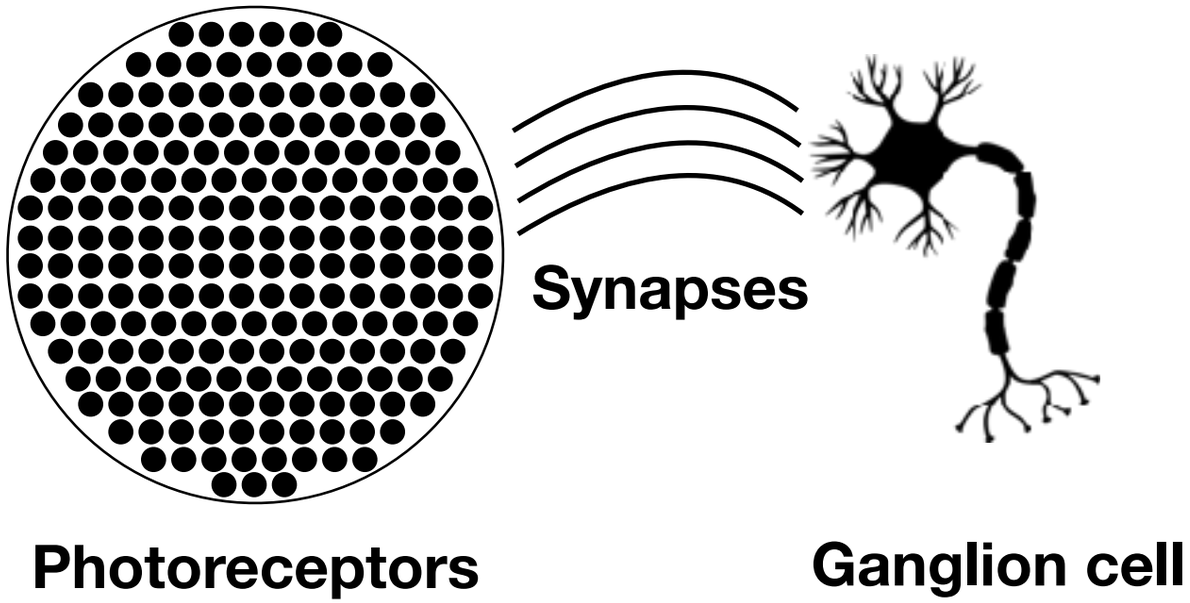

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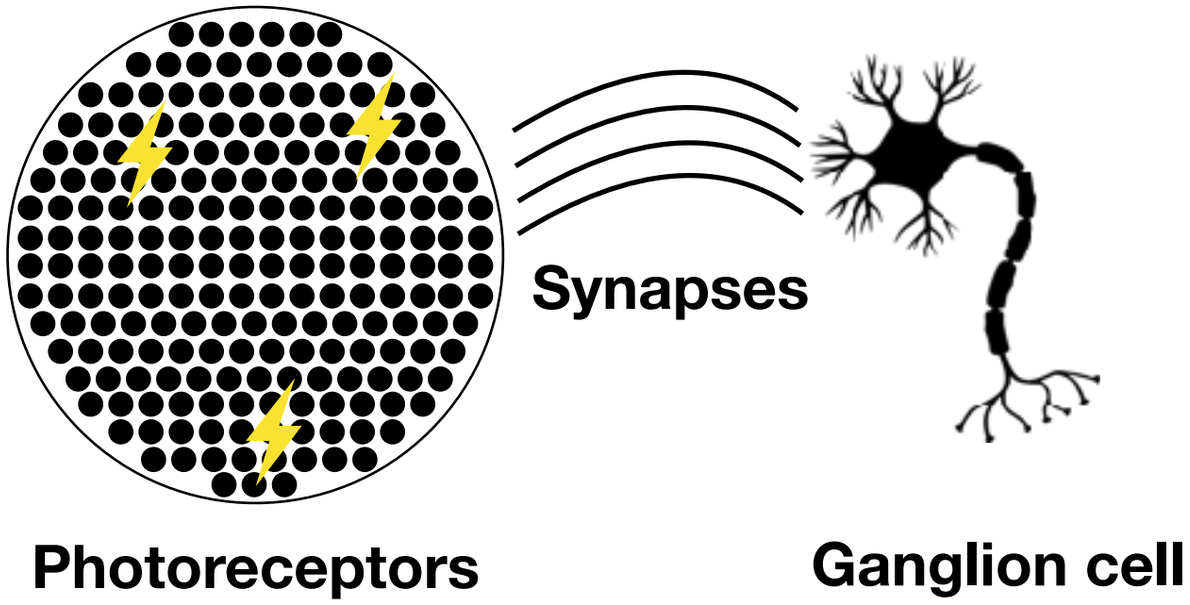

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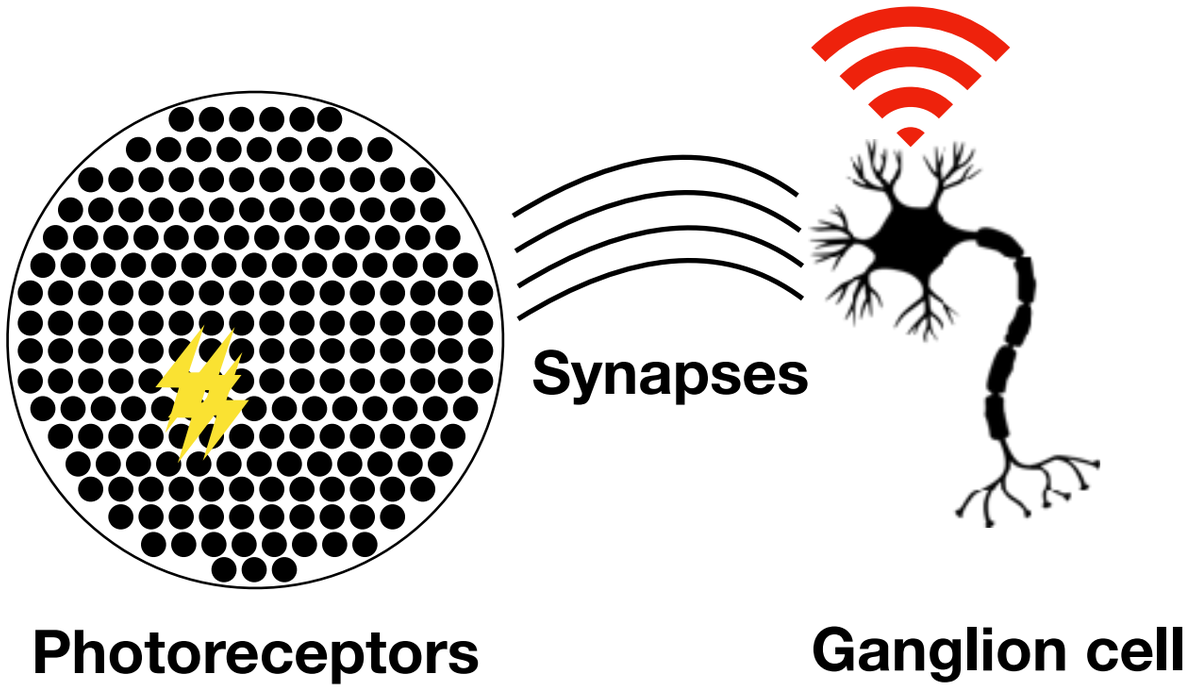

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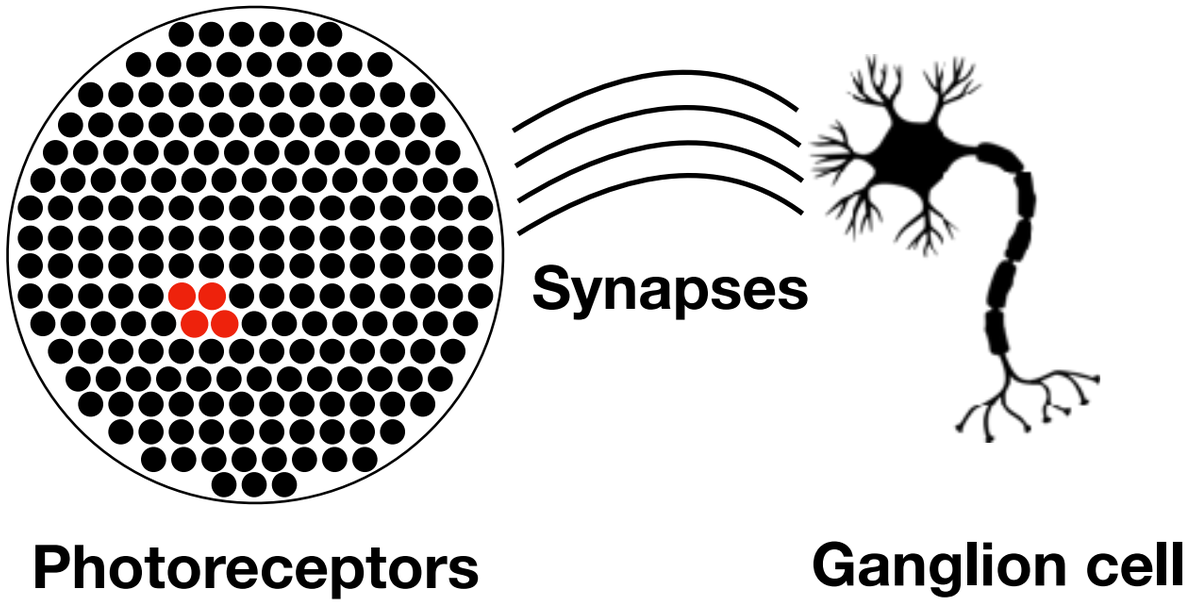

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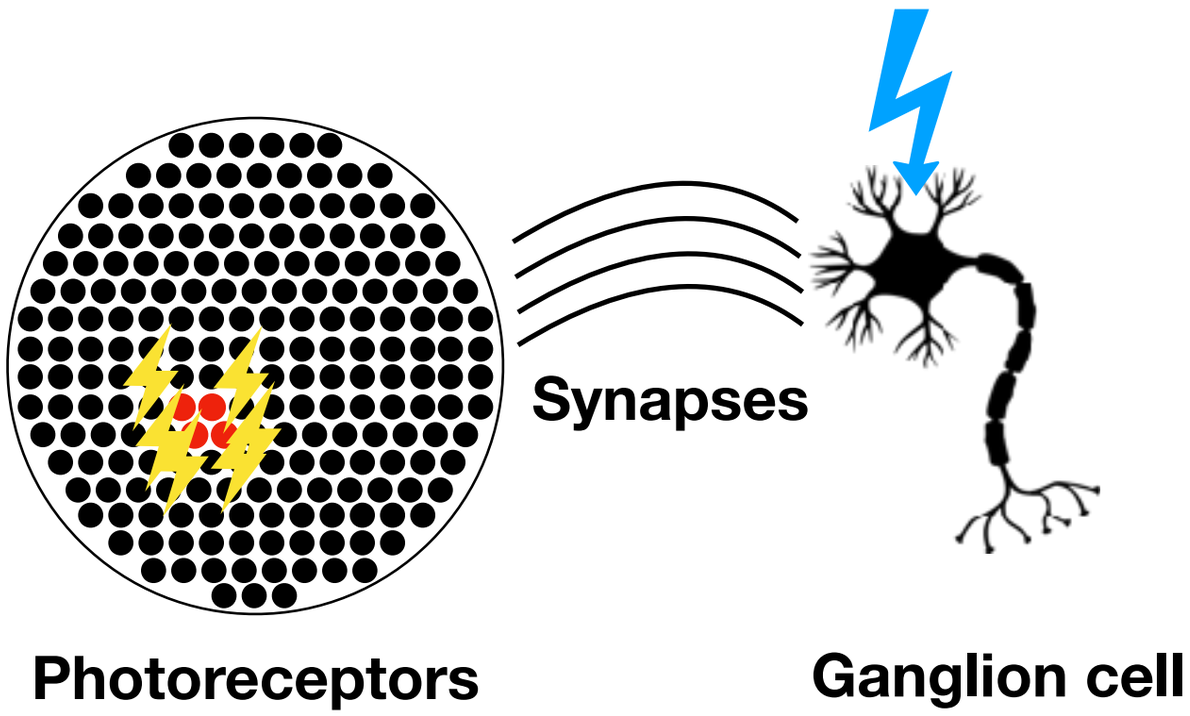

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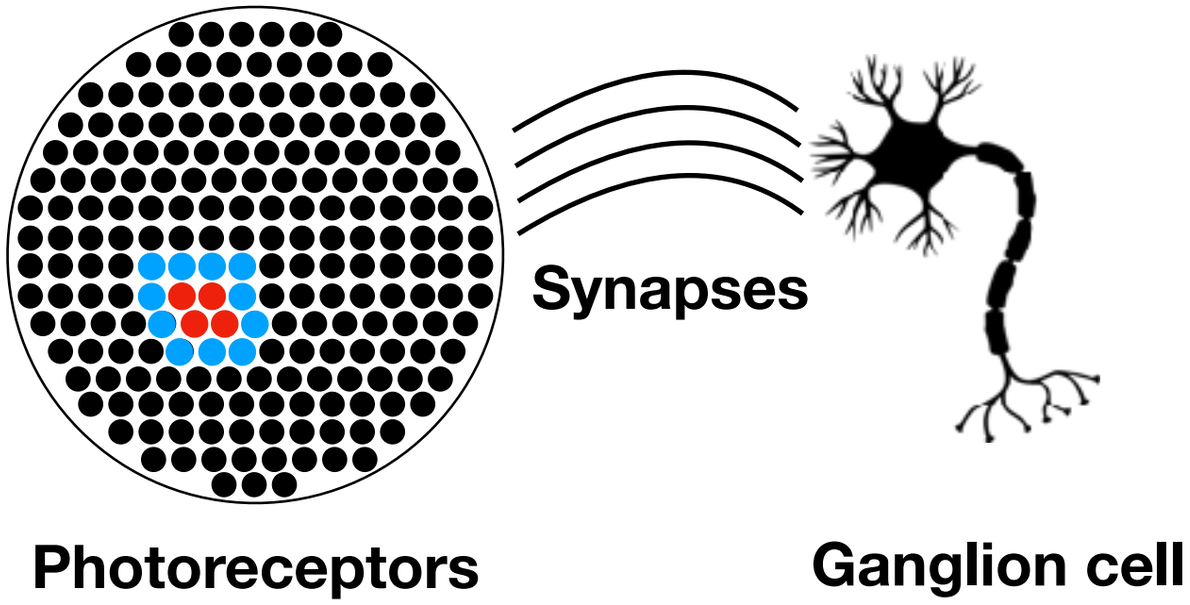

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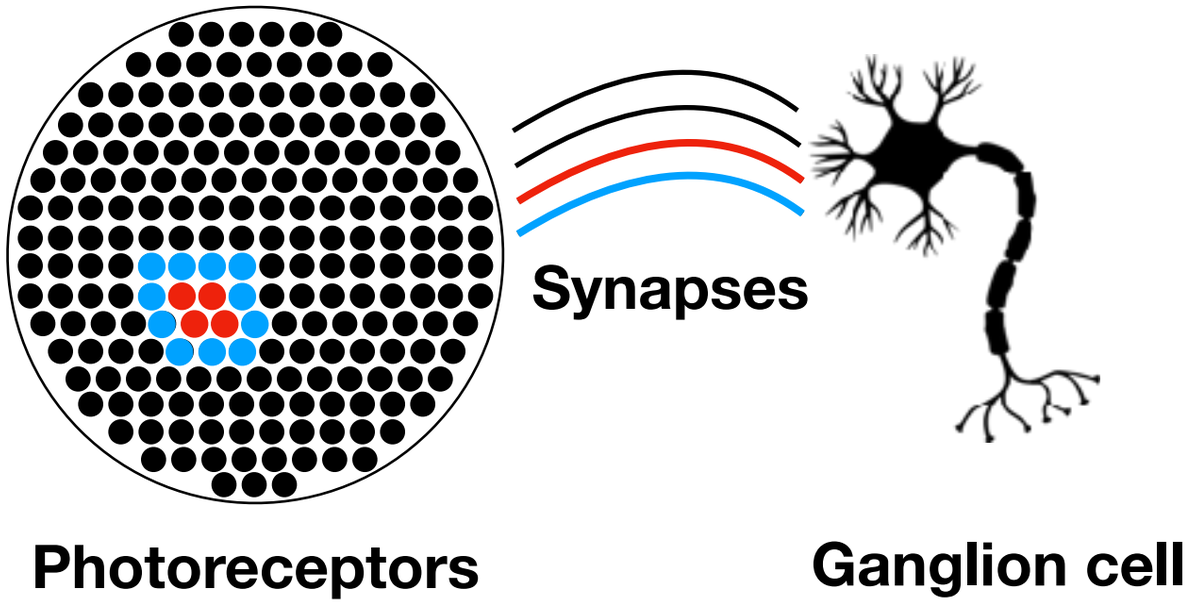

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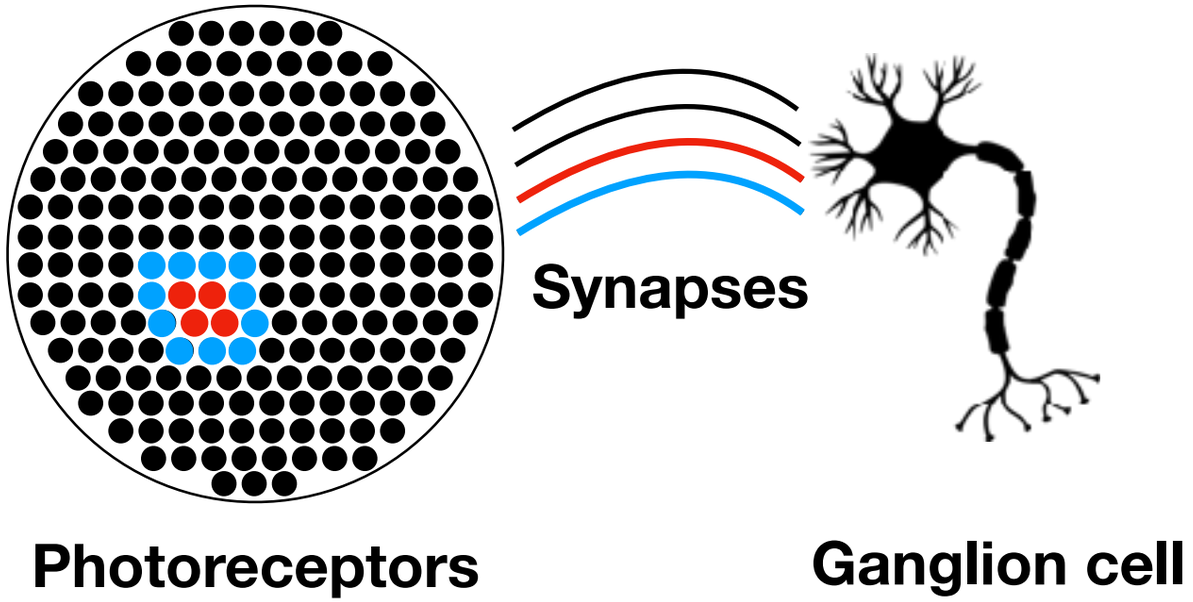


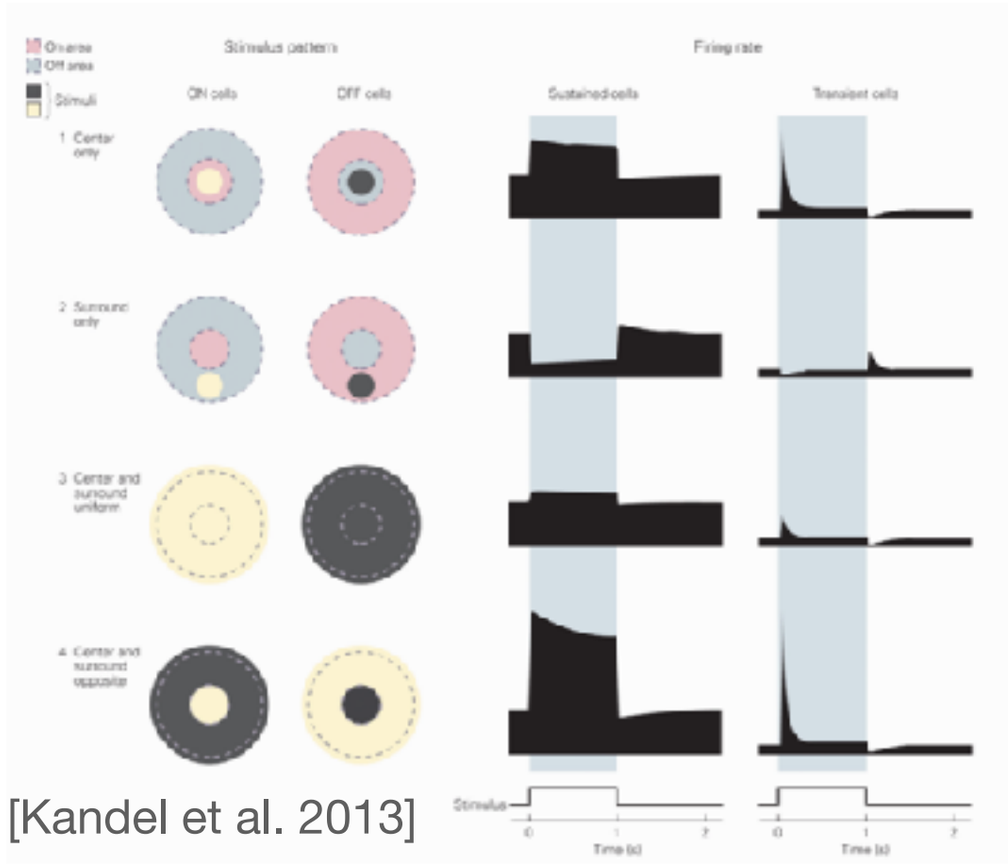


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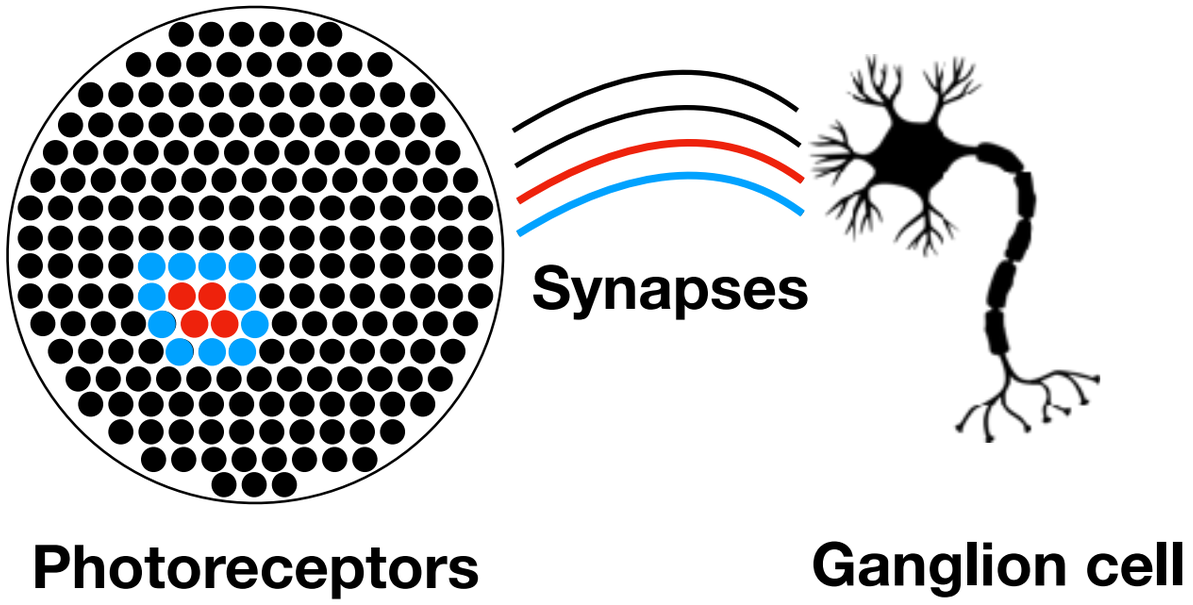


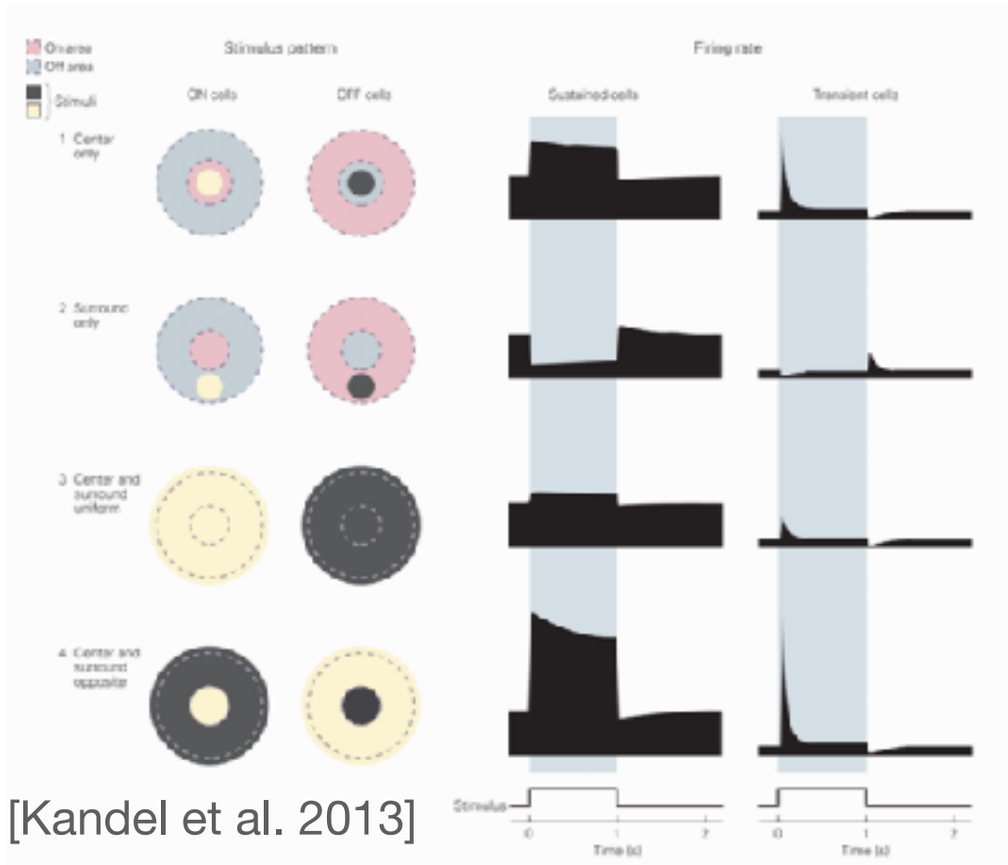


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- A signature of a ganglion cell is the center-surround receptive field.
- [Atick, Redlich 1990] Principal component analysis (PCA) induces center-surround receptive fields and maximize information.

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**How does the retina implement streaming PCA?**

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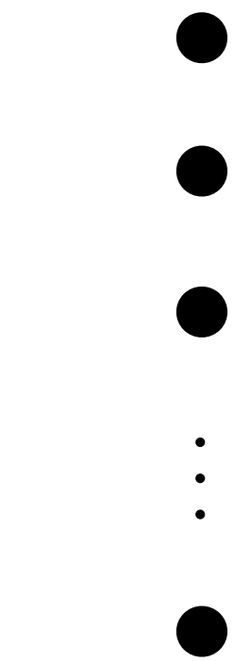

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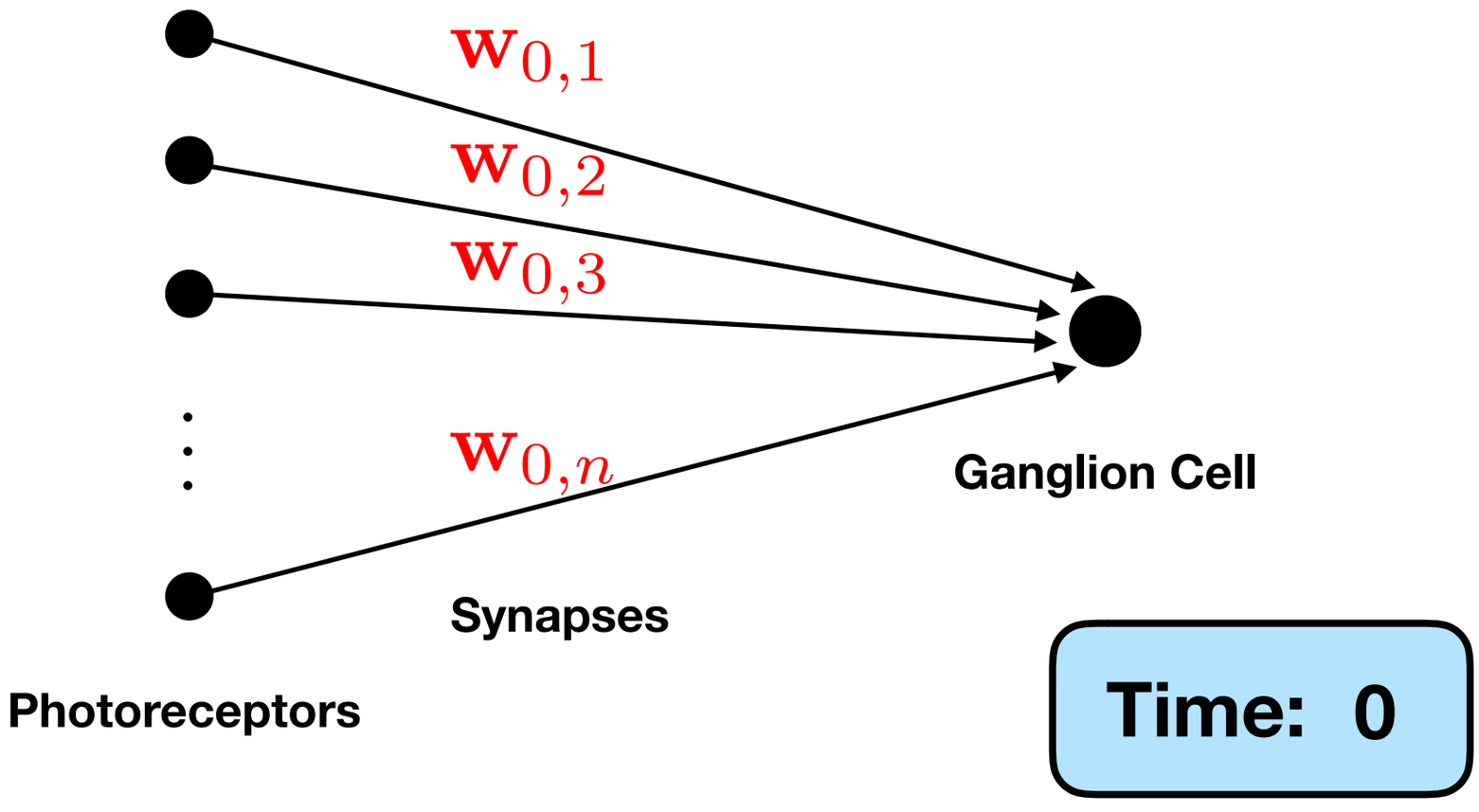

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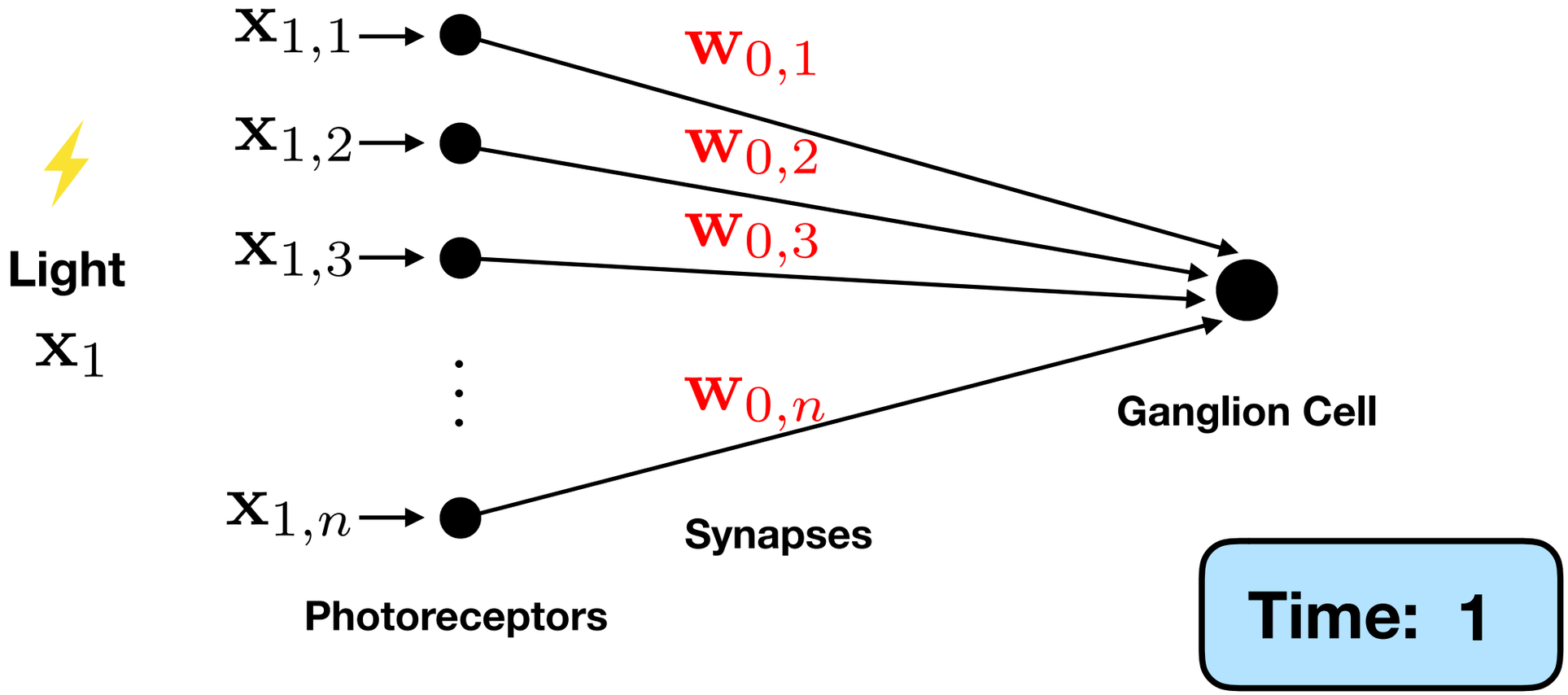

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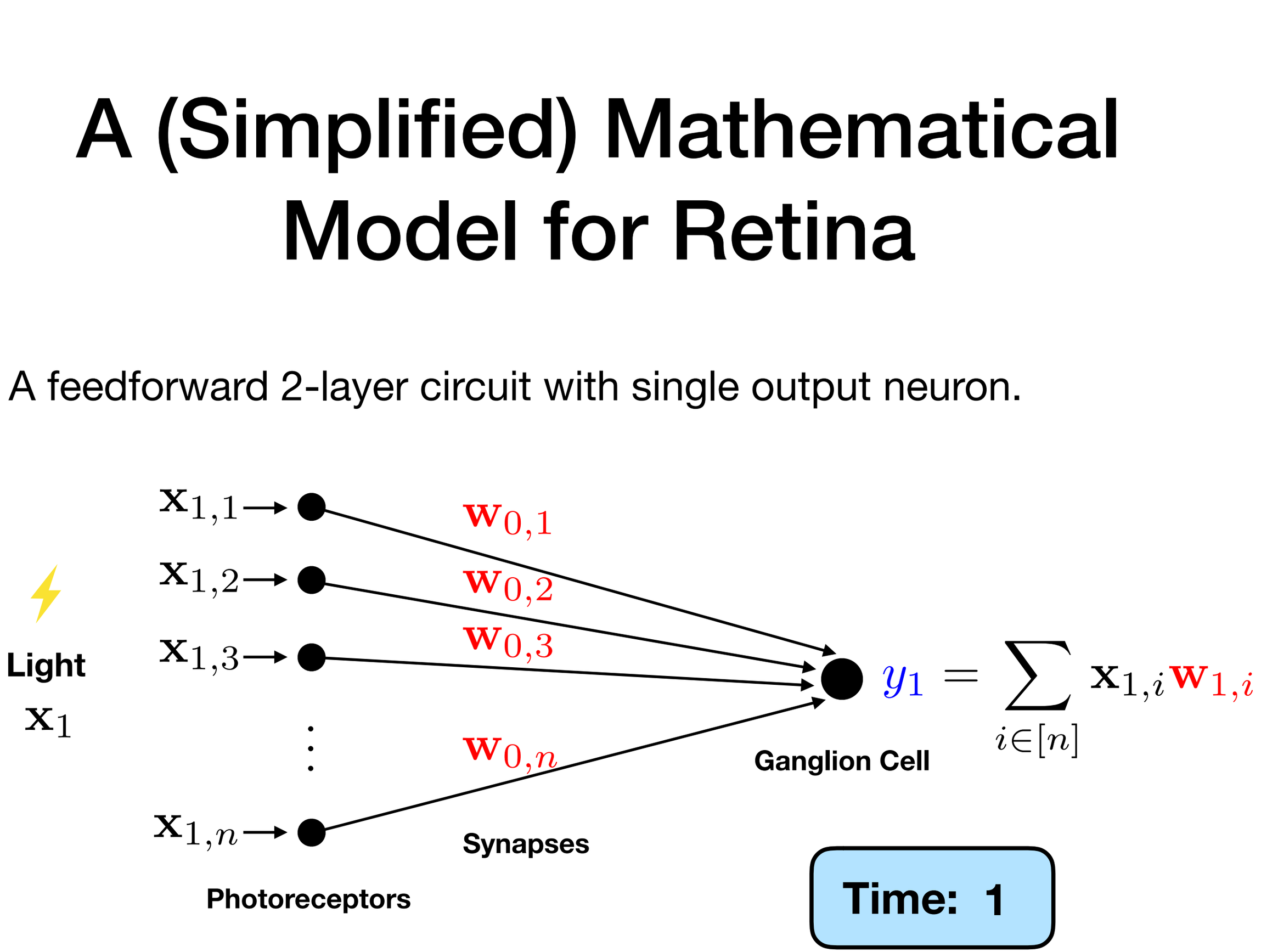

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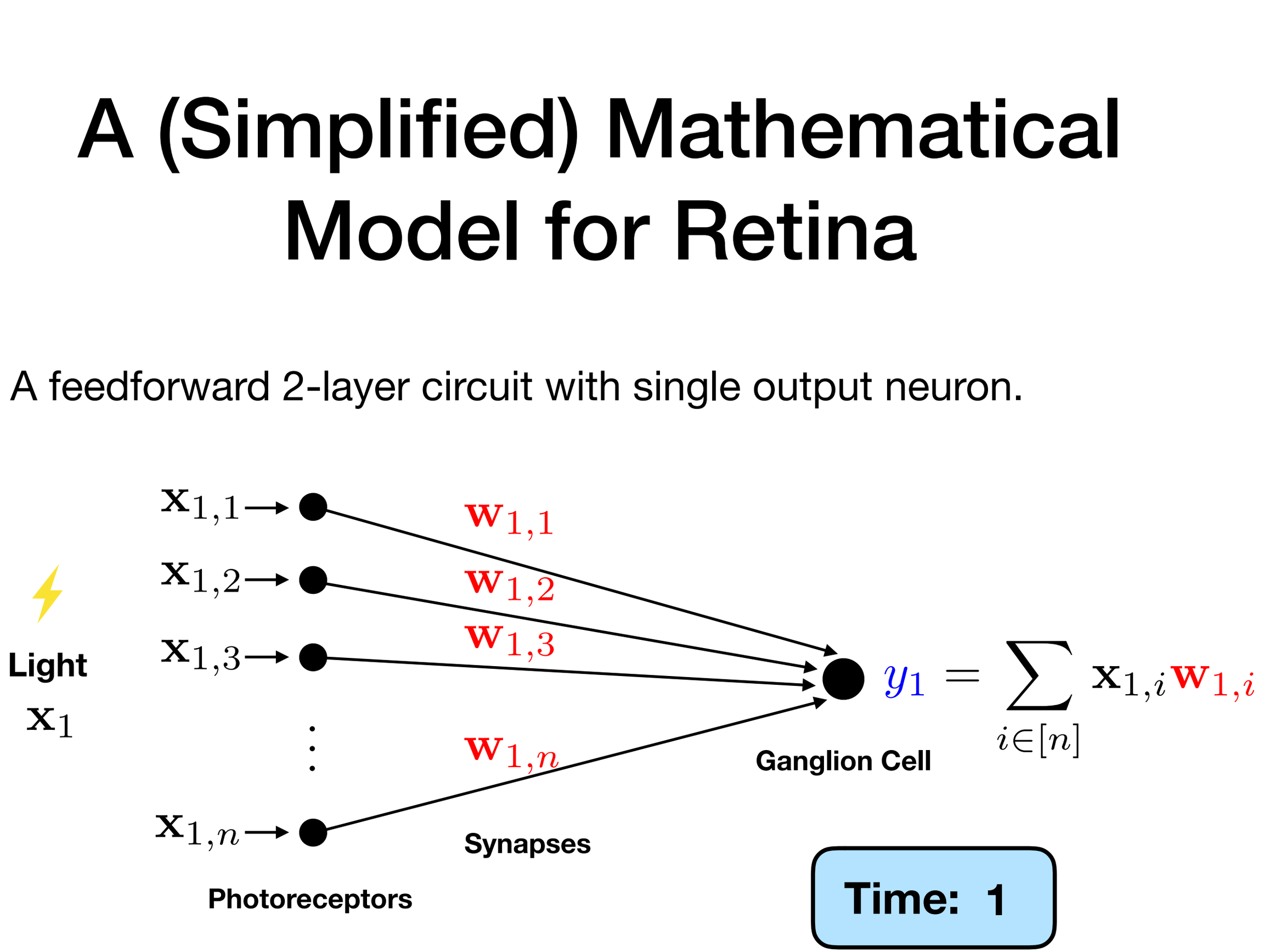

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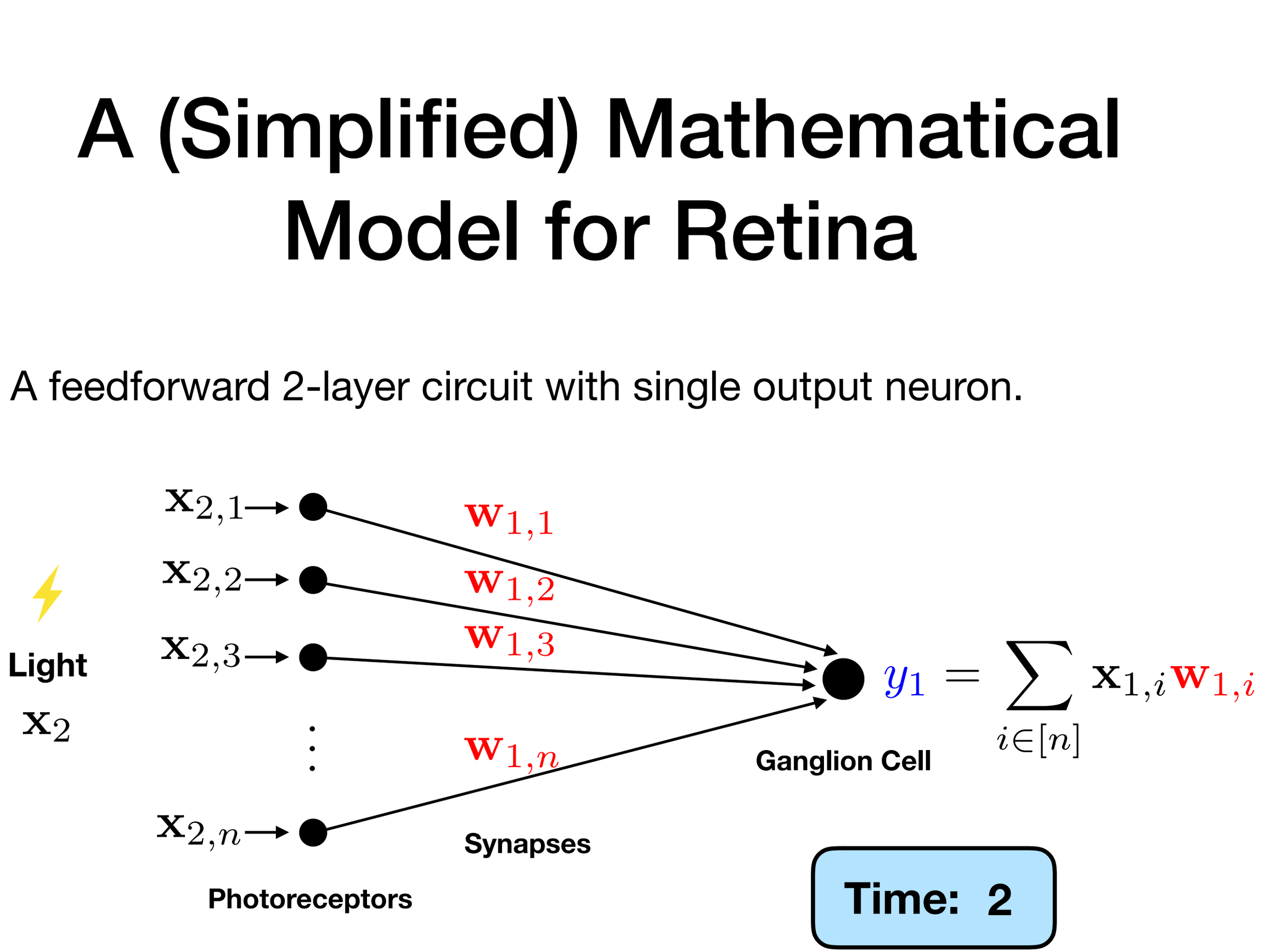

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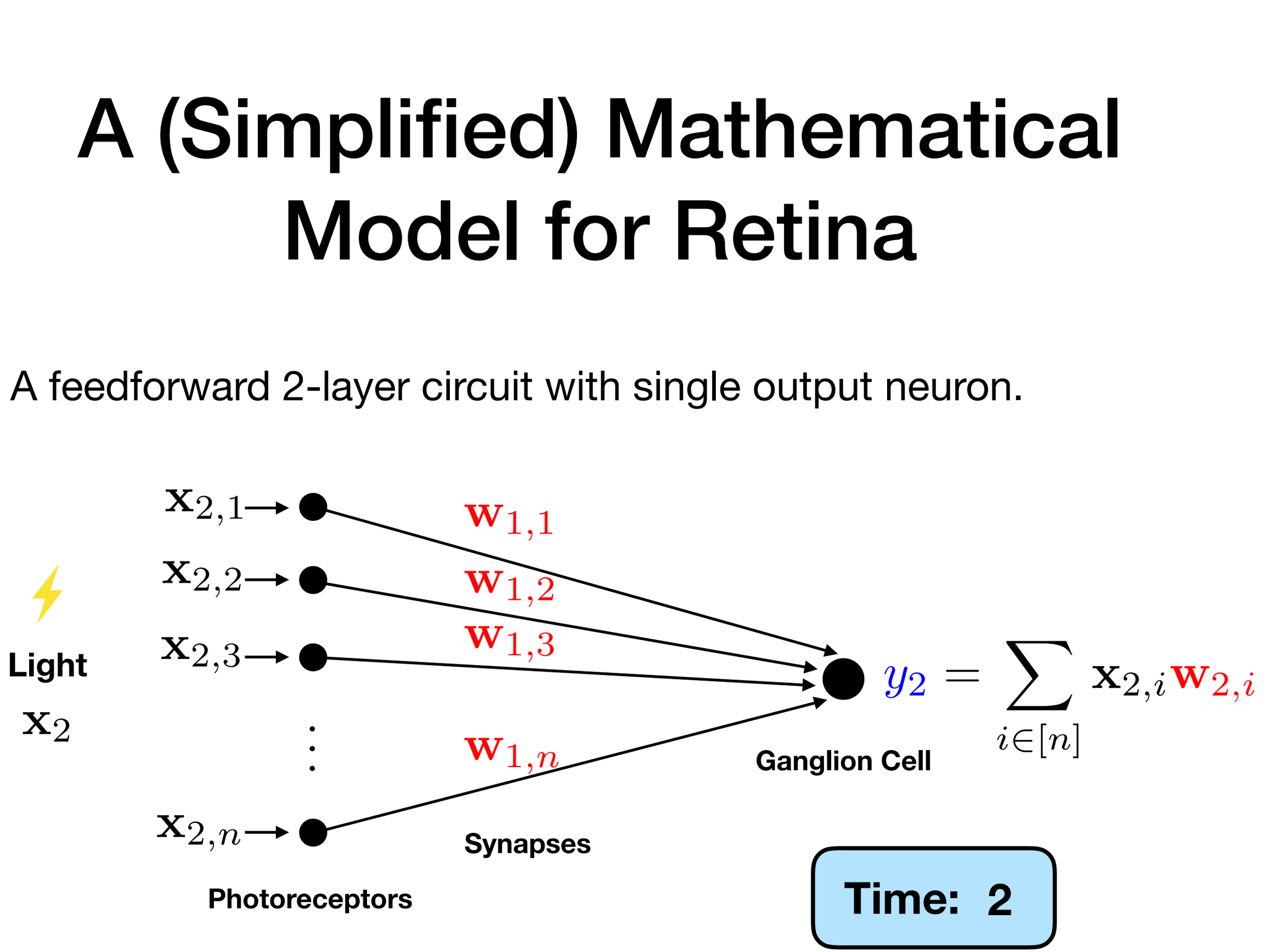

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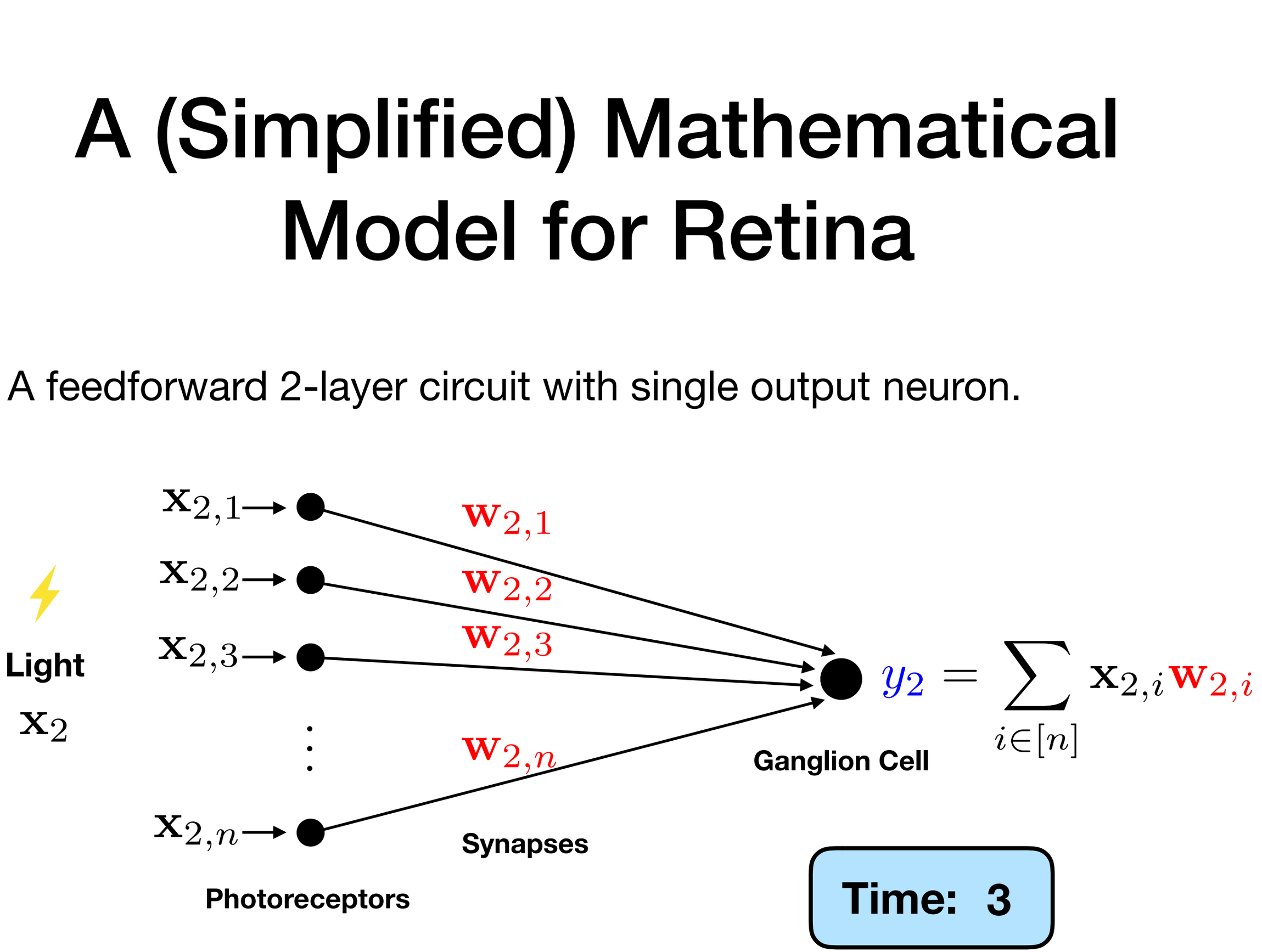

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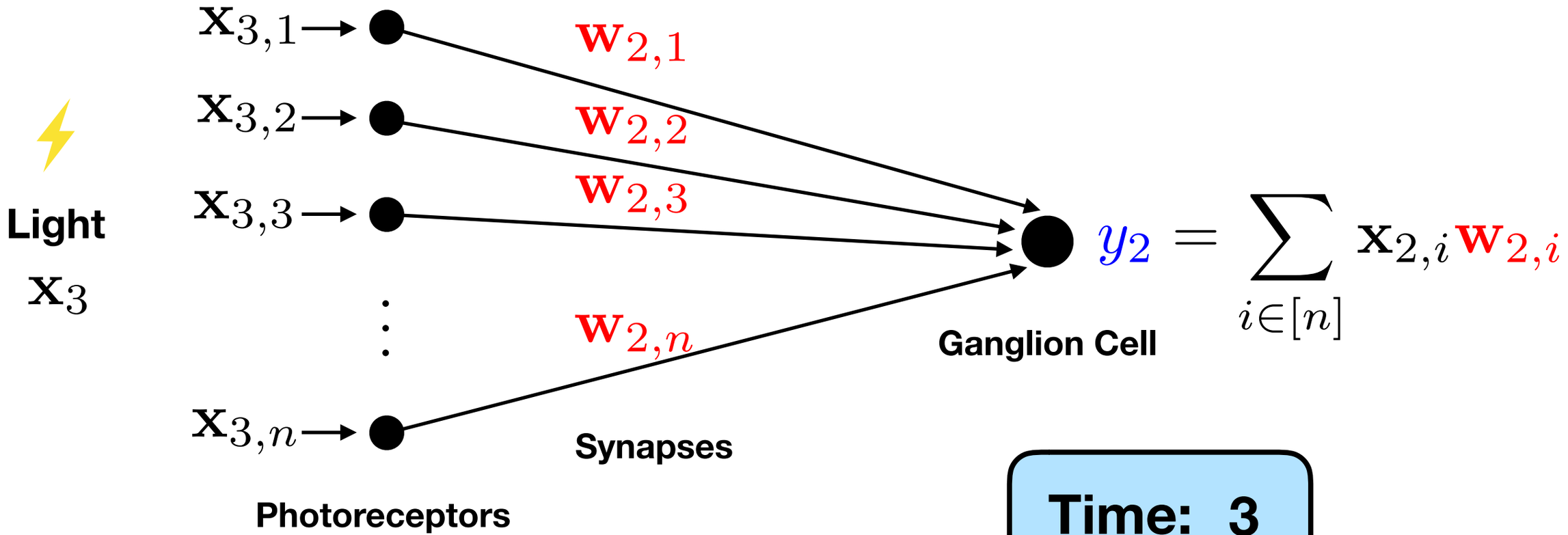

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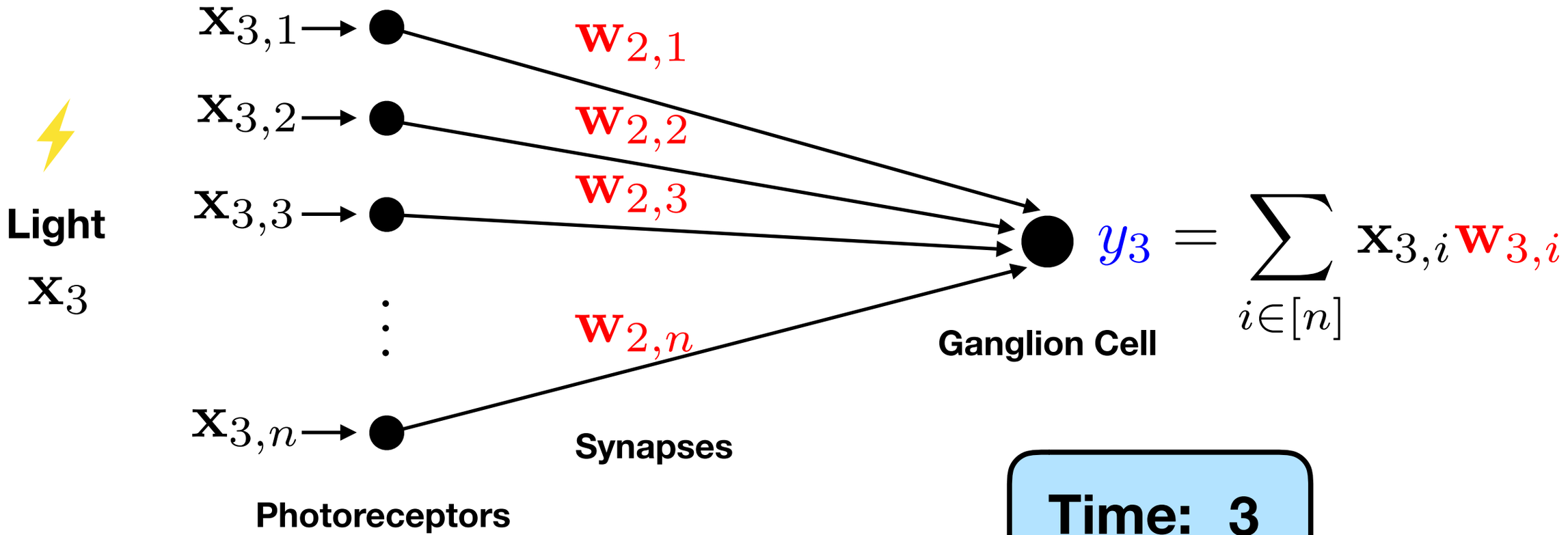

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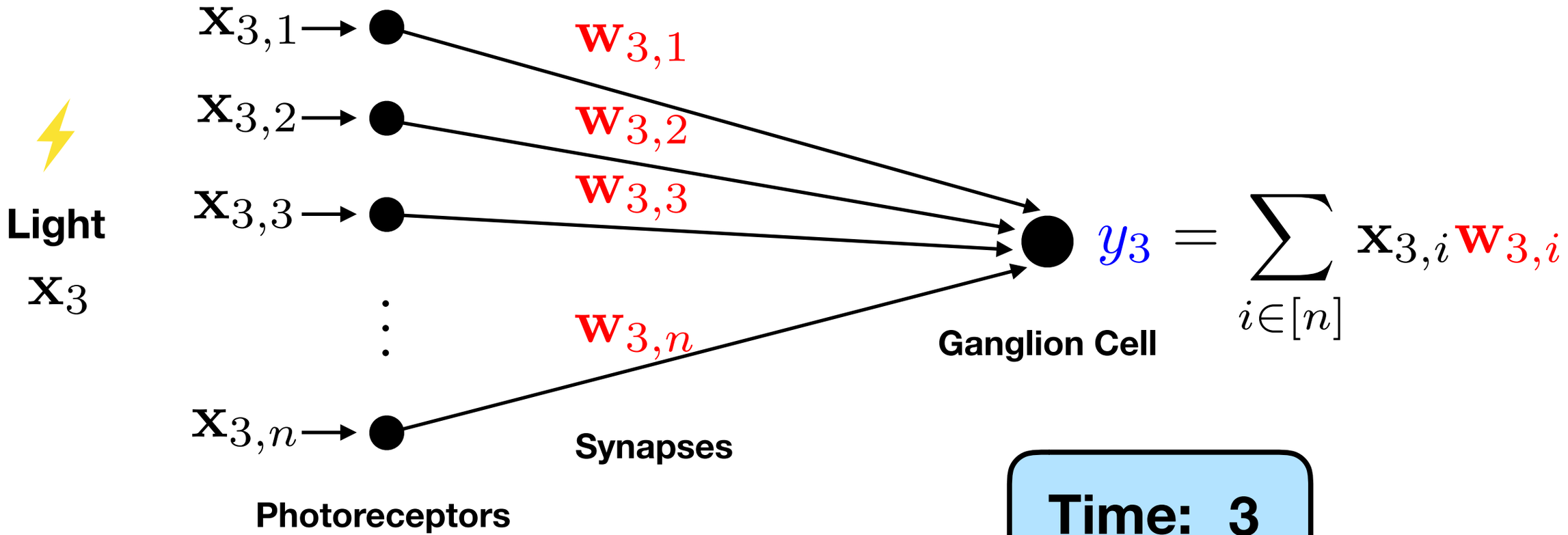

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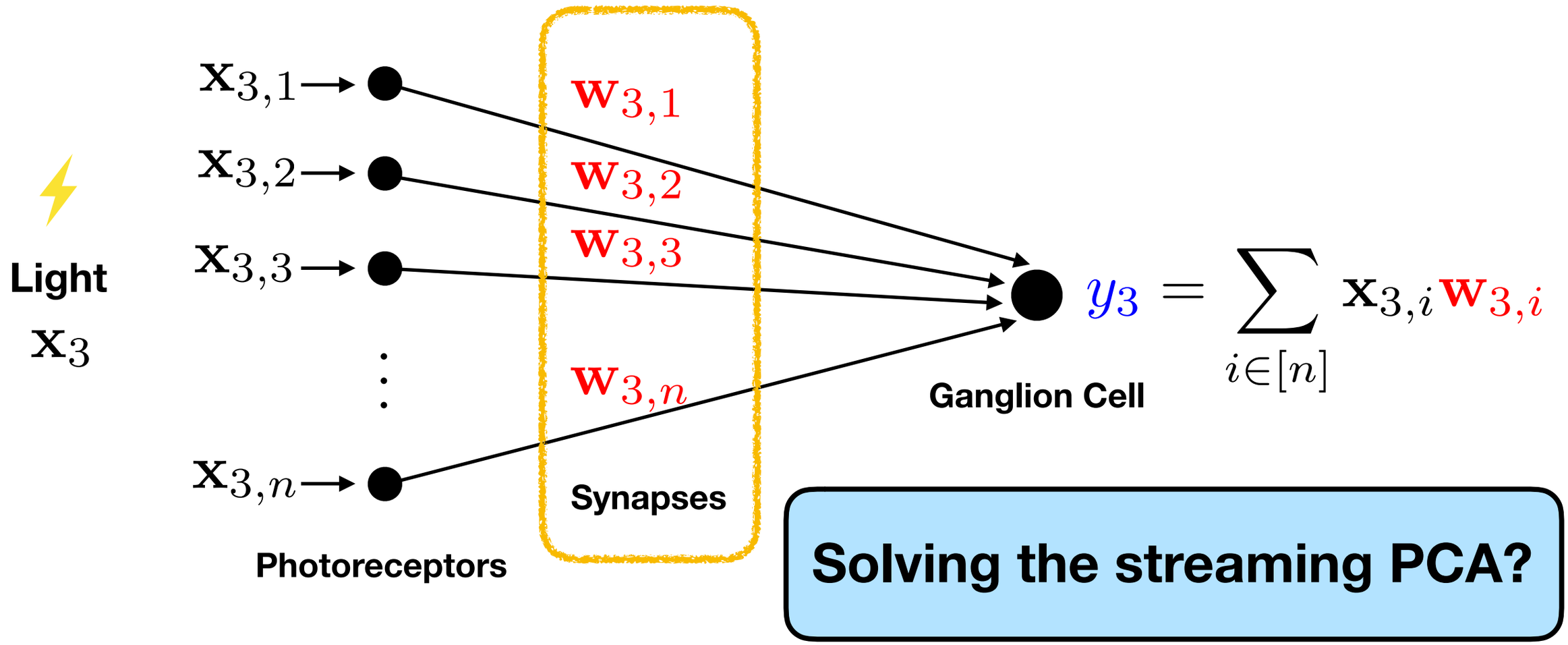


**Solving the streaming PCA?**

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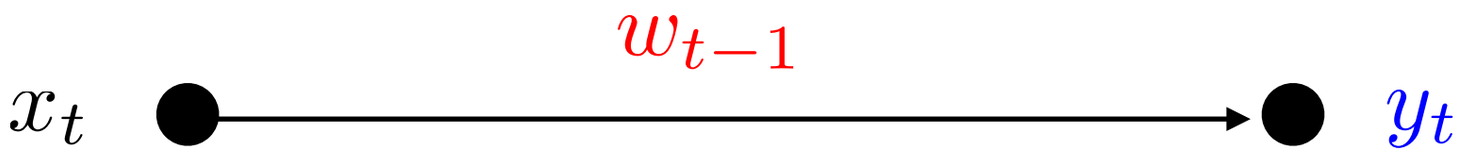

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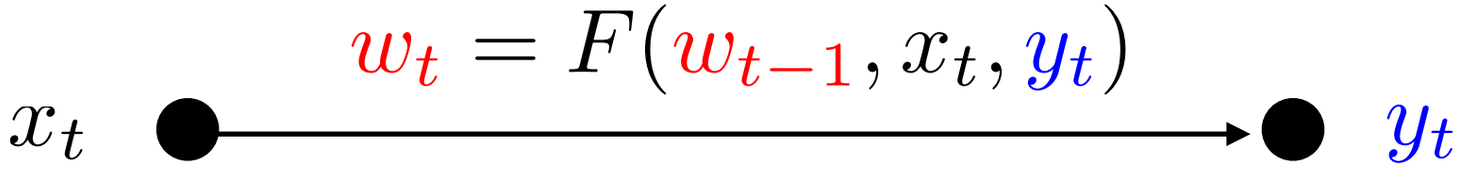

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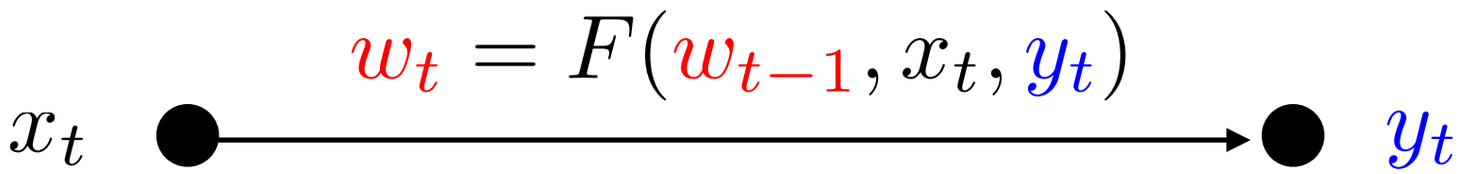


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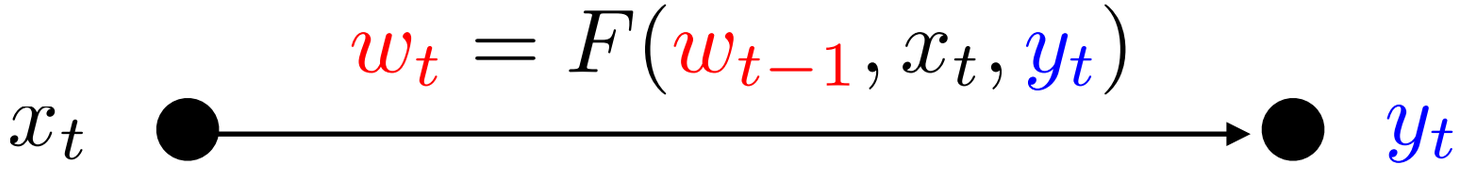


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# What's the Compression Mechanism in Retina?

## A Computational Task

(Capturing Experimental Observation)

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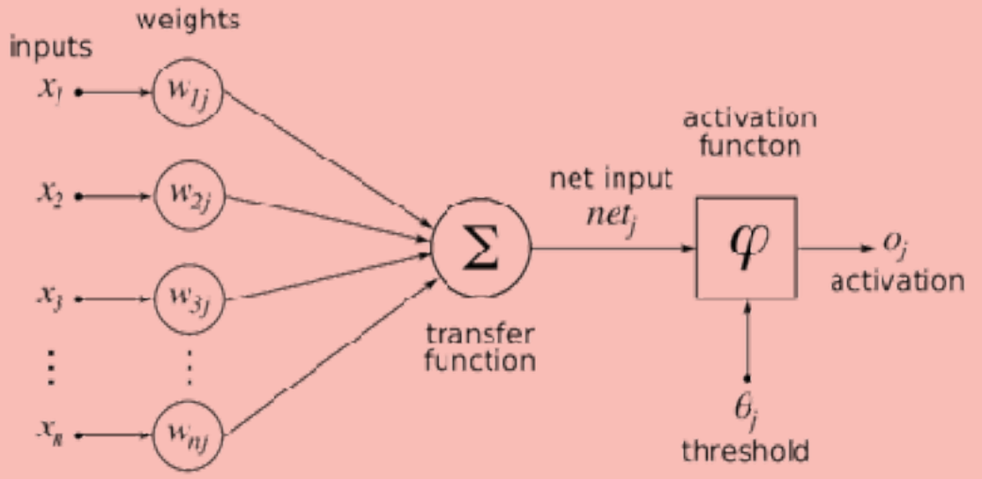
(Subject to biological constraints)

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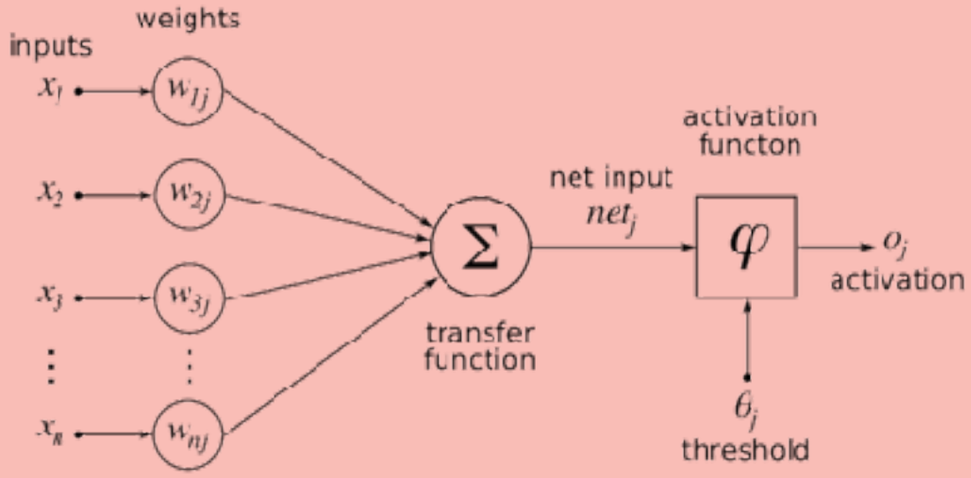


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**Tiny Dependency on #Photoreceptors!**

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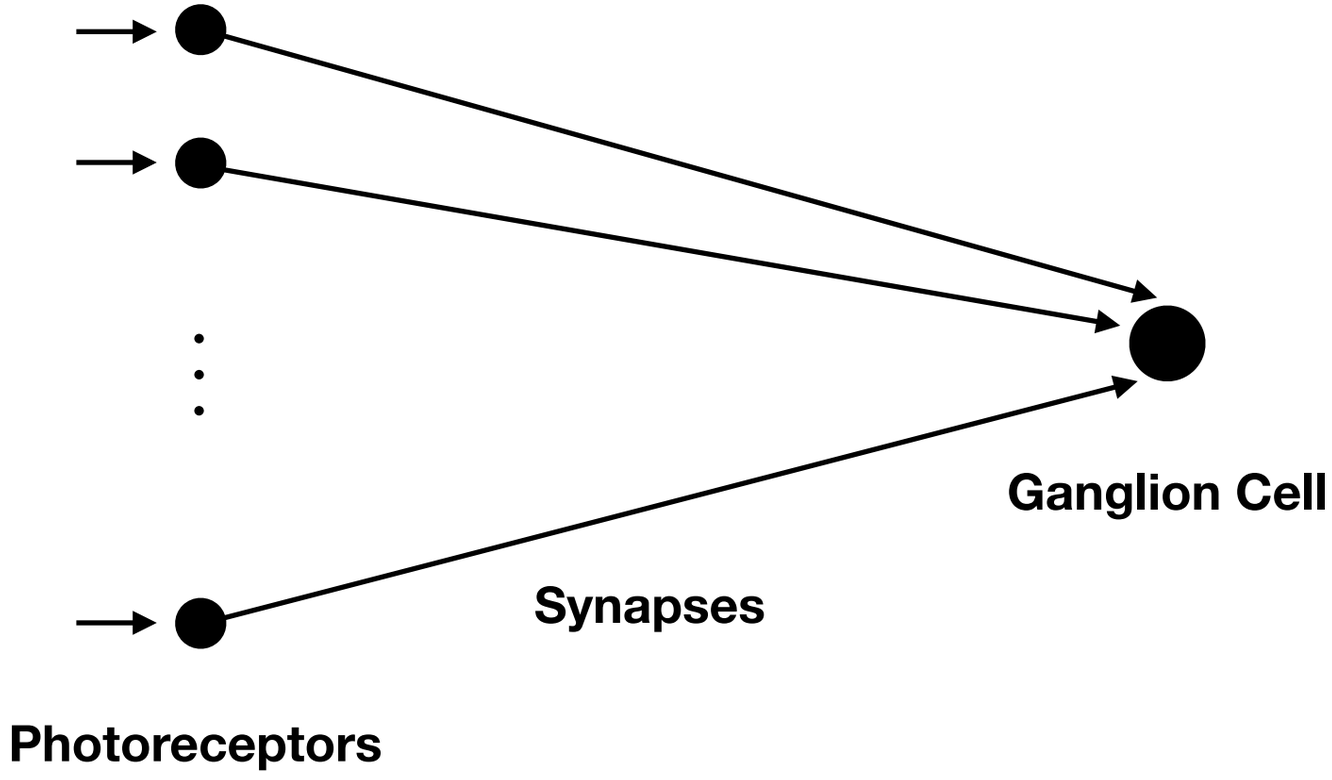

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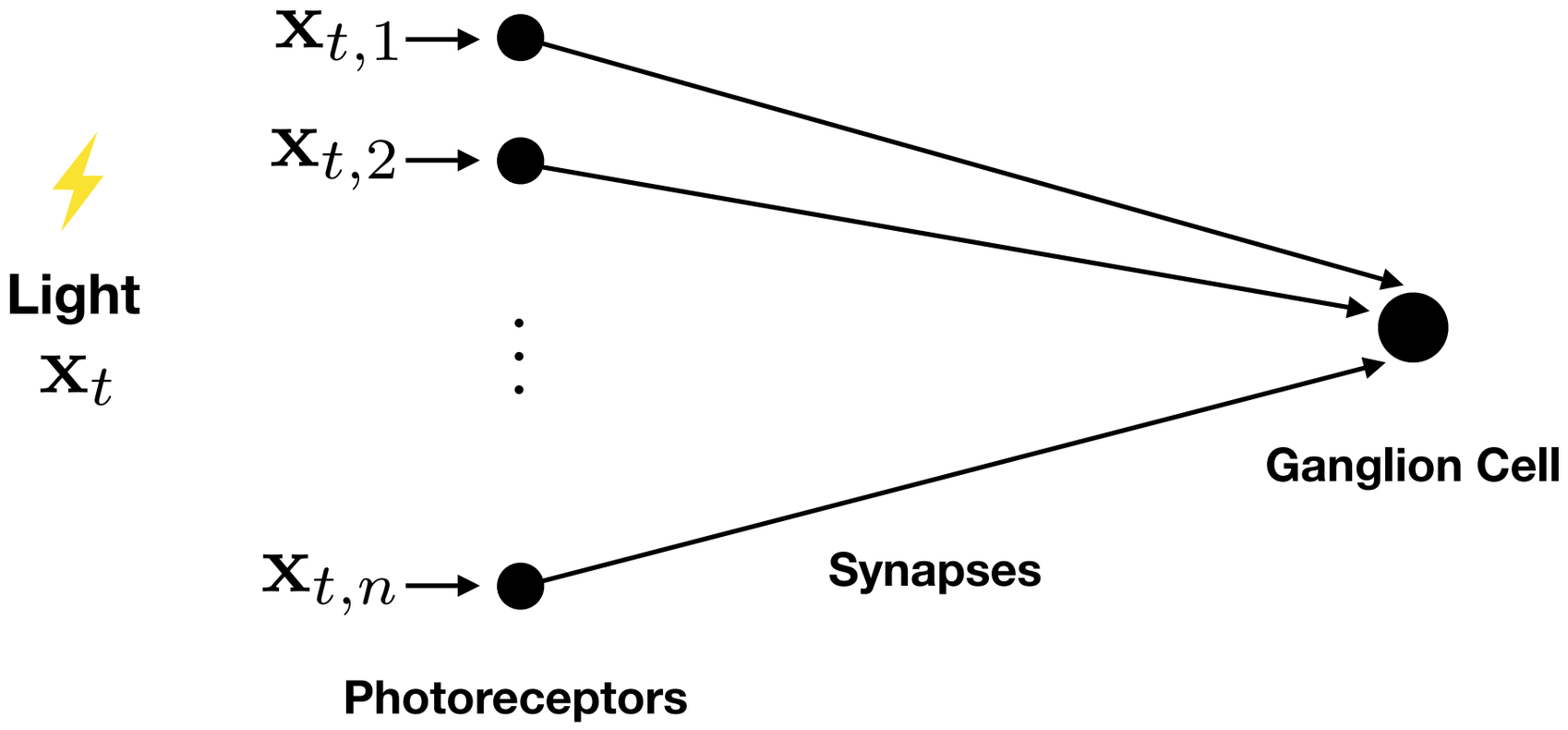

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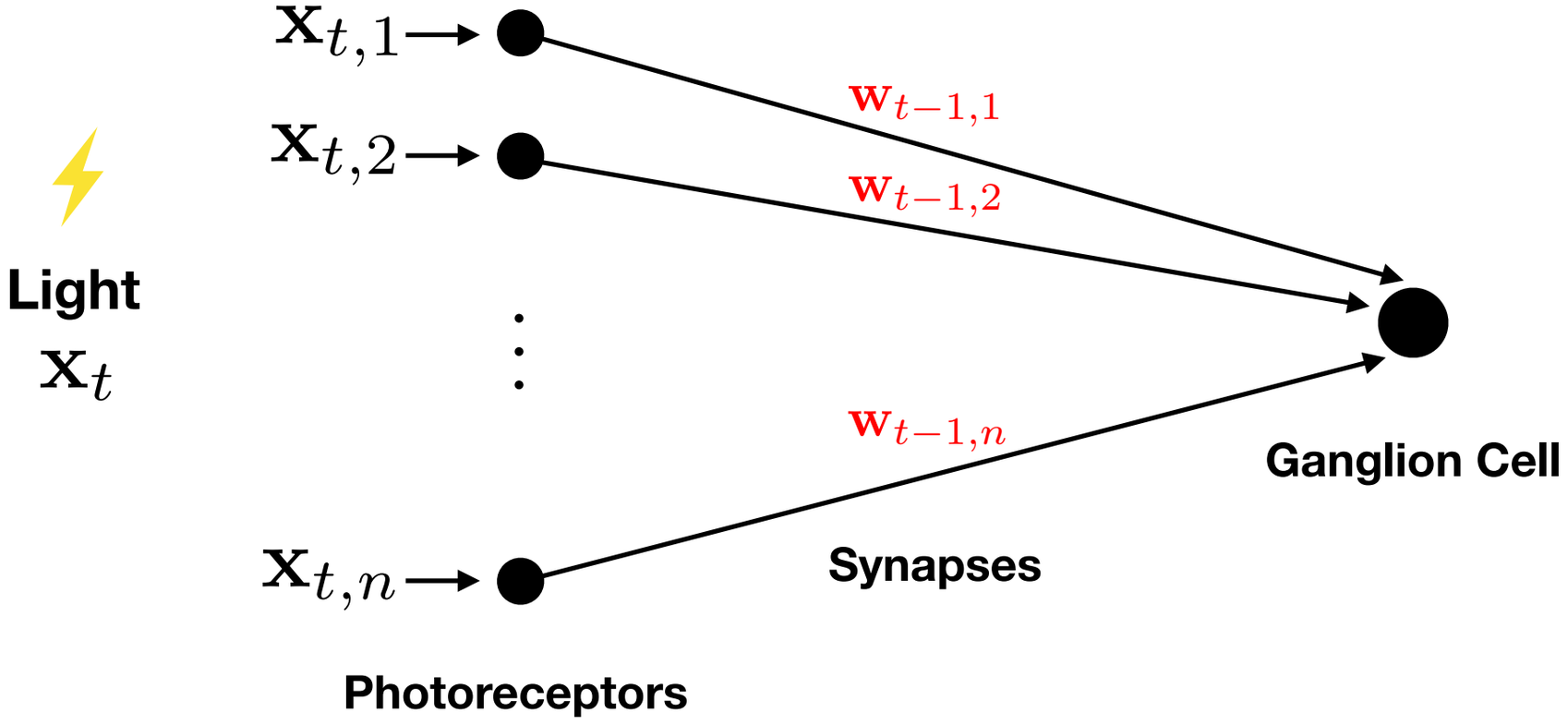

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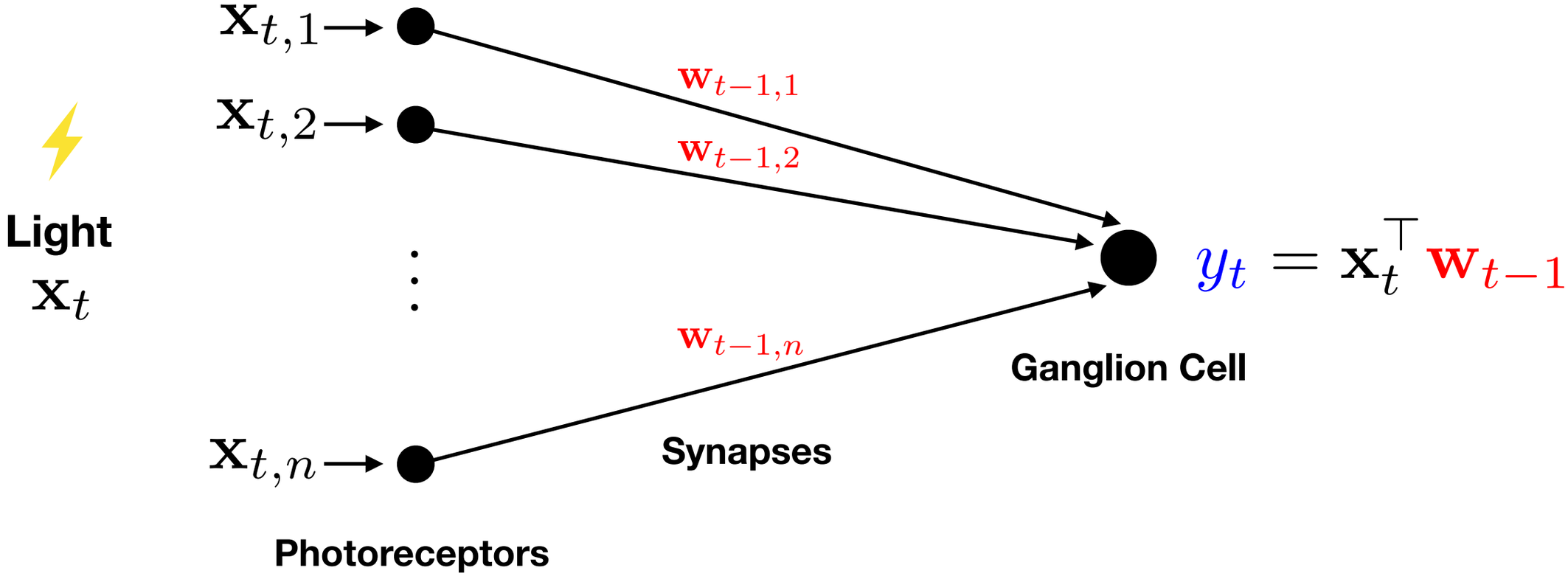

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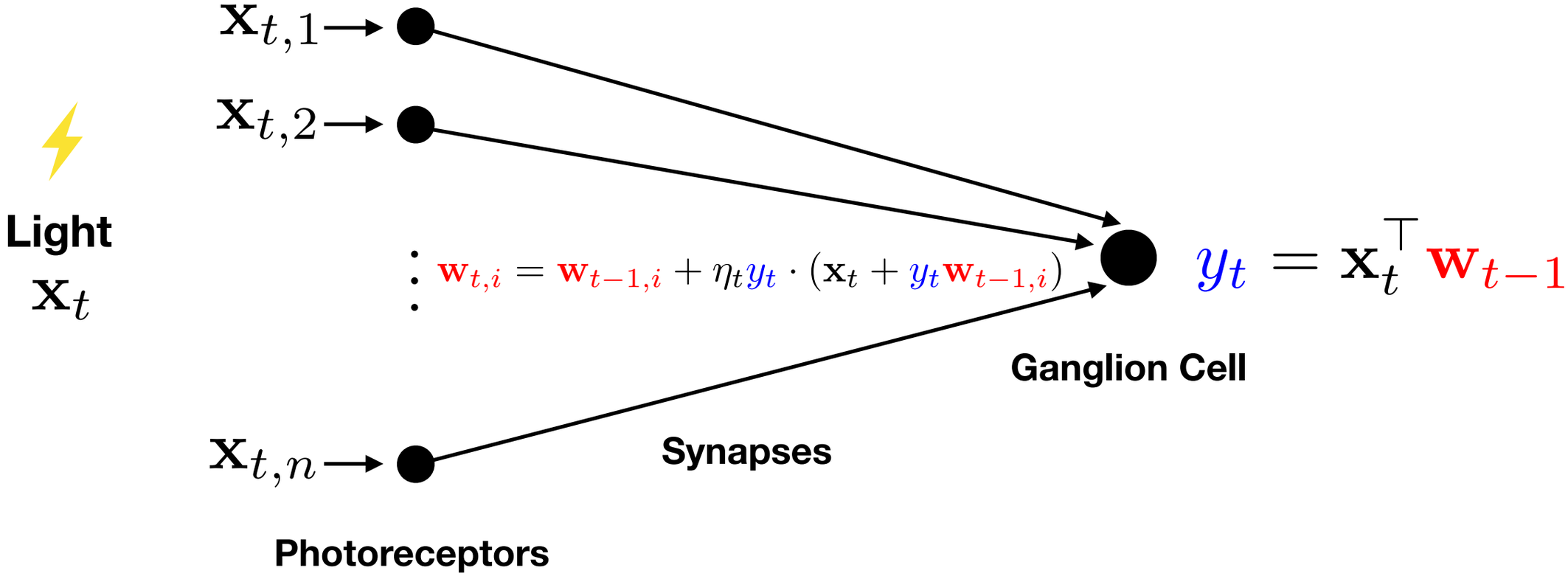

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**Theorem ([Duflo 2013], informal).**

Let $\mathbf{w}_0$ be a random unit vector and $\{\eta_t\}$ be the learning rate. If (i) $\sum_t \eta_t = \infty$ and (ii) $\sum_t {\eta_t}^2 < \infty$, then

$$\lim_{t \to \infty} \langle \mathbf{w}_t, \mathbf{v}_1 \rangle^2 = 1$$

almost surely where $\mathbf{v}_1$ is the normalized top eigenvector of the covariance matrix.

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**Theorem (Oja's rule efficiently solves streaming PCA).**

Let $\mathbf{w}_0$ be a random unit vector and $\epsilon, \delta > 0$. There exist learning rate $\{\eta_t\}$ with $\eta_t = \Theta(1/\log t)$ and

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where $\lambda_1, \lambda_2$ are the top two eigenvalues, such that

$$\Pr\left[\exists t \geq T,\ \frac{\langle \mathbf{w}_t, \mathbf{v}_1 \rangle^2}{\|\mathbf{w}_t\|_2^2} < 1 - \epsilon\right] < \delta\,.$$

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**This improves the best known result on streaming PCA!**

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- The learning rate remains large and the guarantee is for-all-time.
- Specifically, we have $\sum_t {\eta_t}^2 = \infty$.

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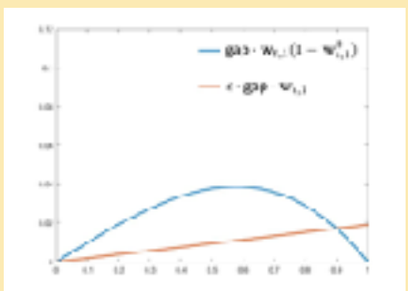

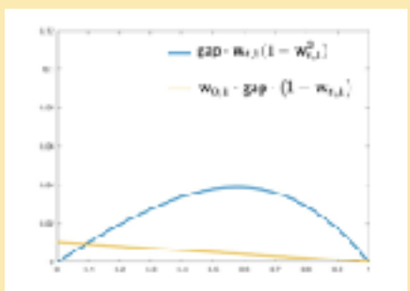

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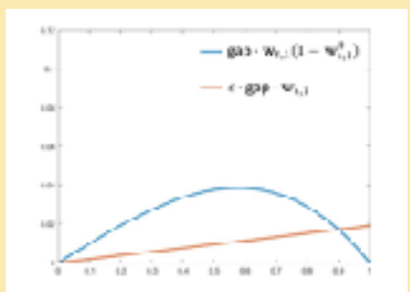

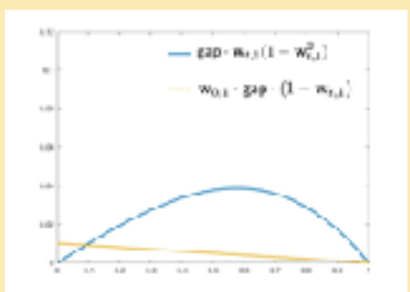

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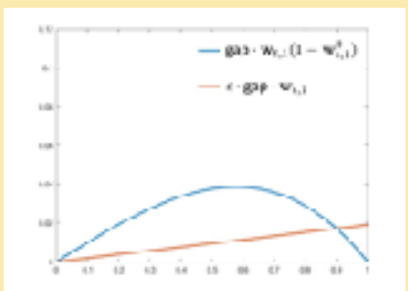

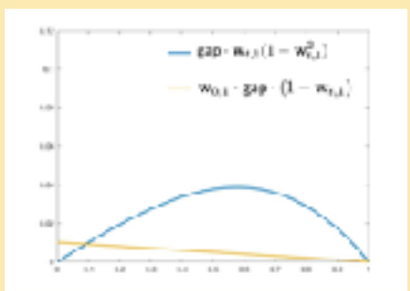

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**Step 1: Linearization & Moment Analysis**

**Step 2: Improvement Analysis**

**Step 3: Interval Analysis**

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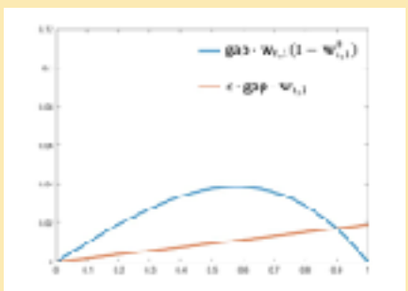
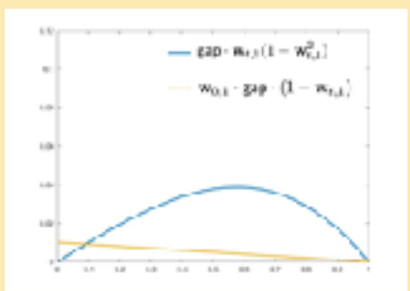




## Stopping Time Framework

*Flexible and nearly optimal!*

**Step 1: Linearization & Moment Analysis**

**Step 2: Improvement Analysis**

**Step 3: Interval Analysis**

See the the full version [arXiv 1911.02363v2] or a follow-up paper [arXiv 2006.06171] focusing on the proof framework/techniques!

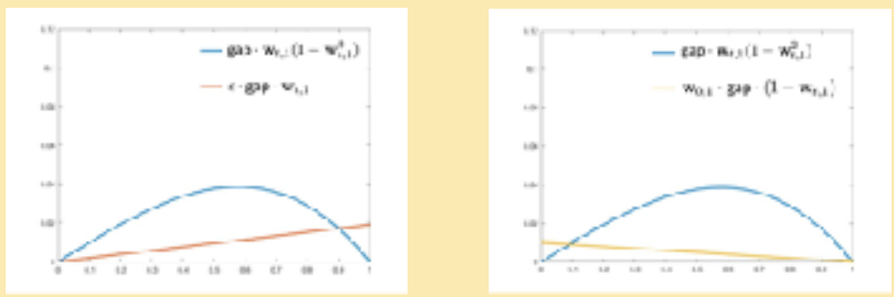
# Few Words on the Proofs

The higher order term makes the previous analysis (for other streaming PCA algorithm) fail in analyzing Oja's rule 😢

$$\mathbf{w}_t = \mathbf{w}_{t-1} + \eta_t y_t \mathbf{x}_t - \eta_t y_t^2 \mathbf{w}_{t-1}$$

## Continuous Analysis

*Suggest the right way to analyze!*



## Stopping Time Framework

*Flexible and nearly optimal!*

**Step 1: Linearization & I ent An sis**

**Step 2: Improvement A**

**Step 3: Interval**

**The technique improves 3 different ML problems!**

See the the full version [arXiv 1911.02363v2] (
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# Future Endeavor

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*When CS & Math Meet Neuroscience!?*

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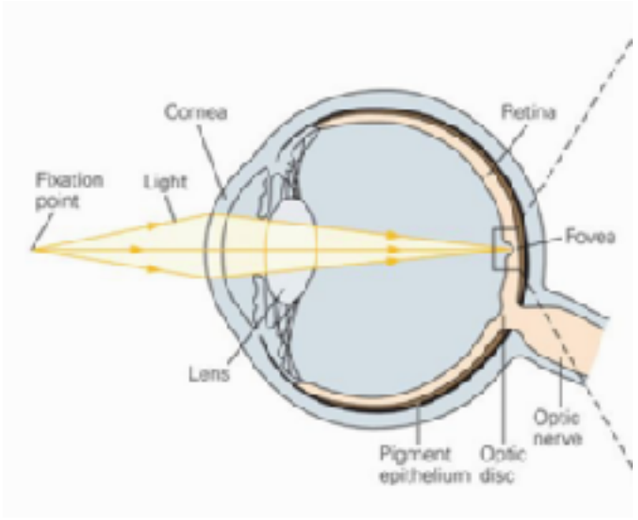
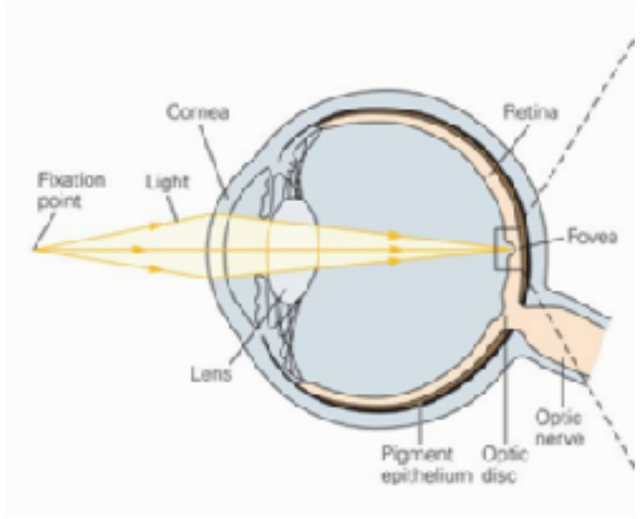
*When CS & Math Meet Neuroscience!?*




**Sensory System**
**Information Theory**

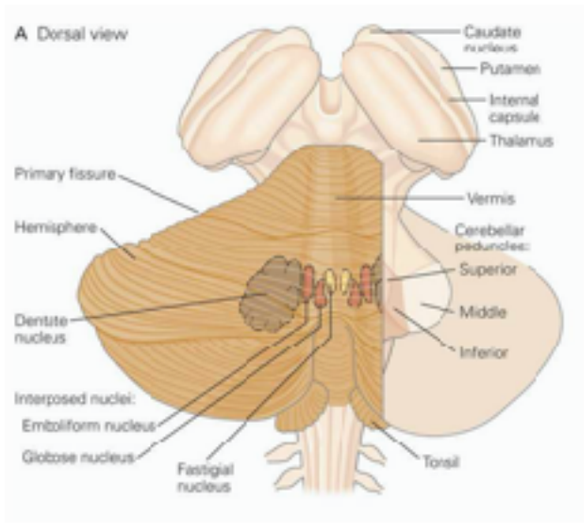
# Future Endeavor

*When CS & Math Meet Neuroscience!?*



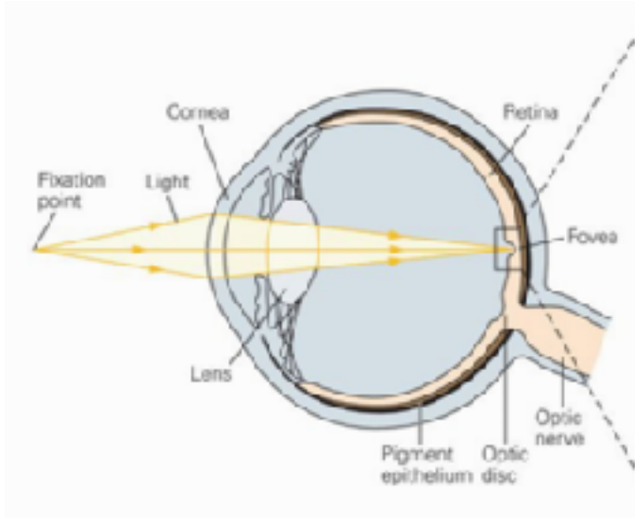

**Sensory System**
**Information Theory**

**Cerebellum**
**Supervised Learning**



# Future Endeavor

*When CS & Math Meet Neuroscience!?*

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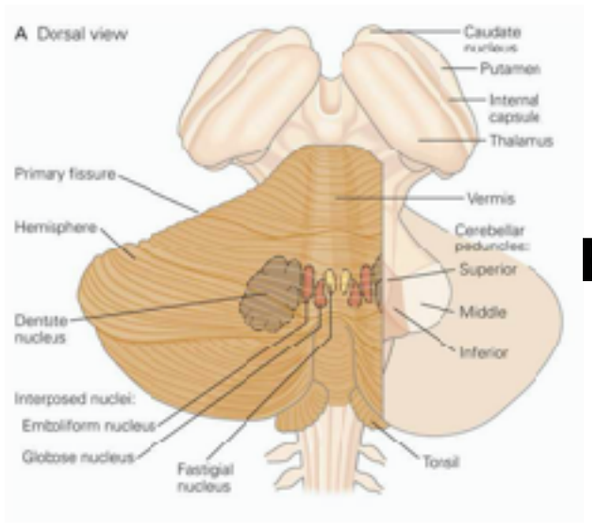
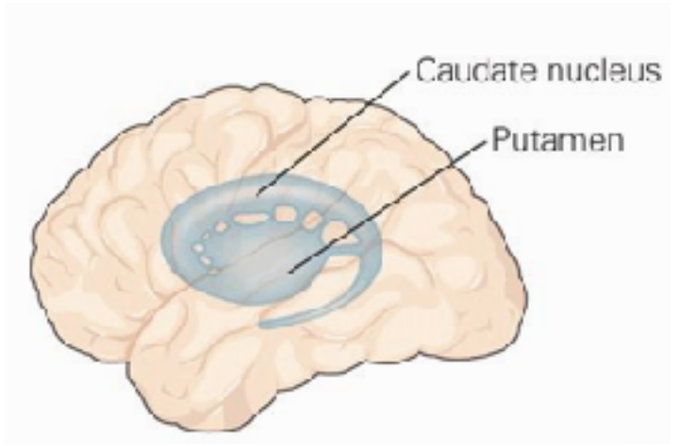
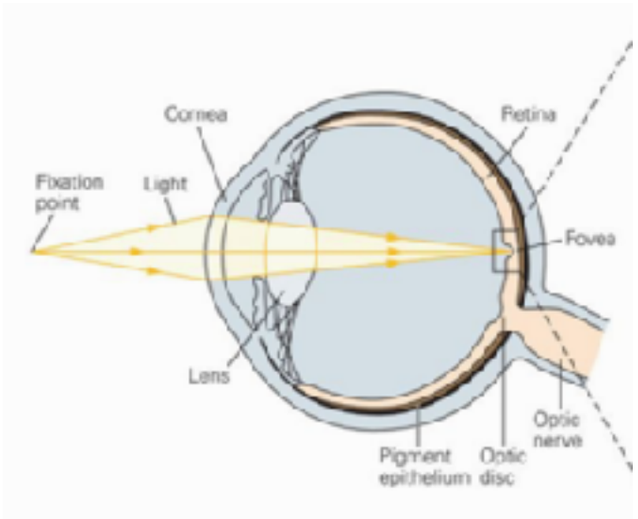



**Basal Ganglia**
**Reinforcement Learning**

# Future Endeavor

*When CS & Math Meet Neuroscience!?*




**Sensory System**
**Information Theory**

**Cerebellum**
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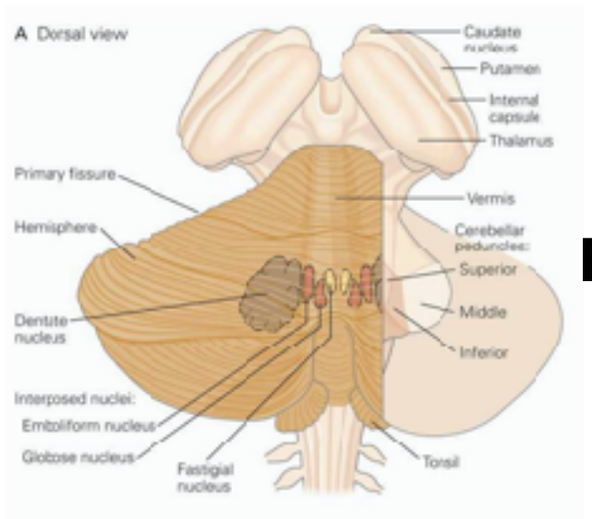
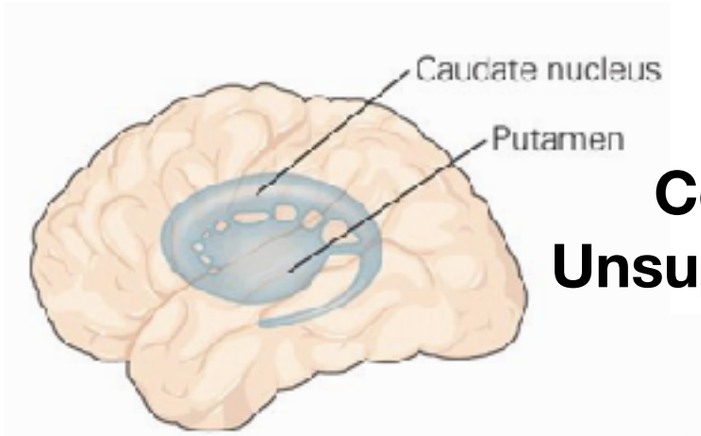
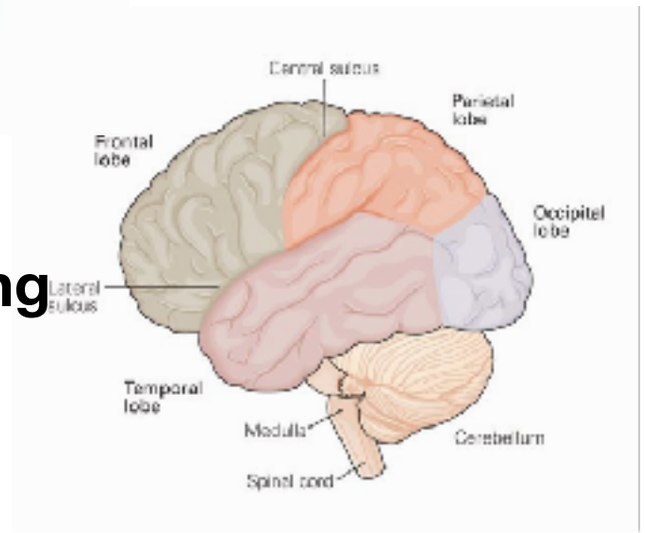


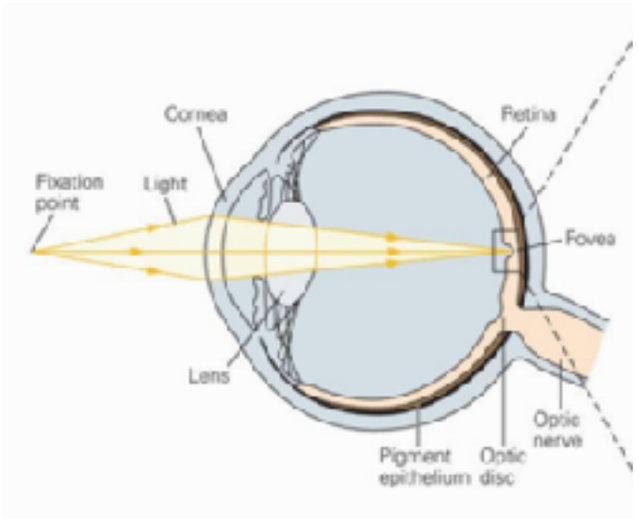

**Basal Ganglia**
**Reinforcement Learning**

**Cerebral Cortex**
**Unsupervised Learning**



# Future Endeavor

*When CS & Math Meet Neuroscience!?*




**Sensory System**
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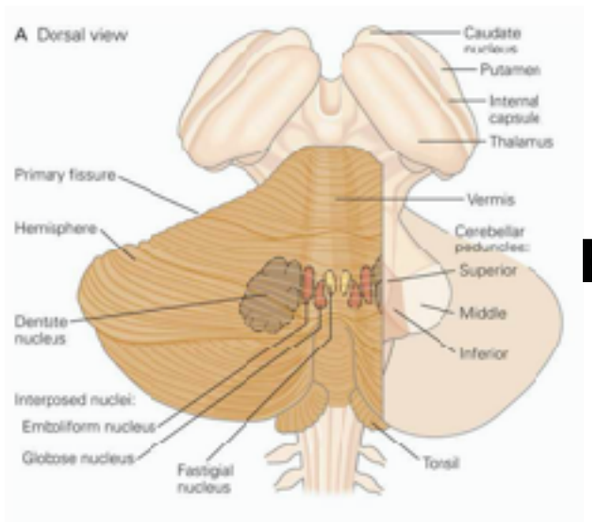


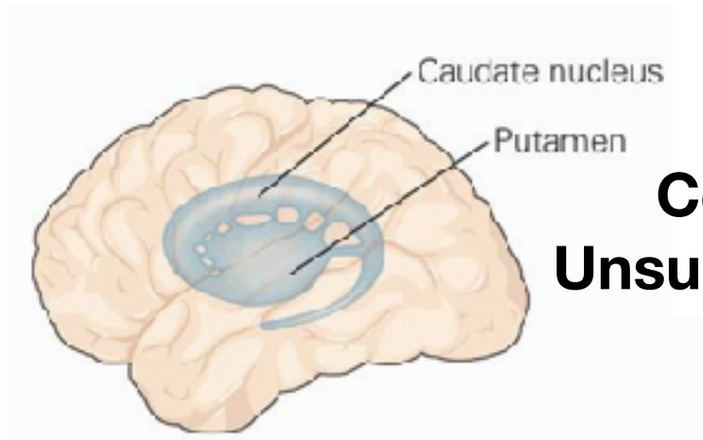


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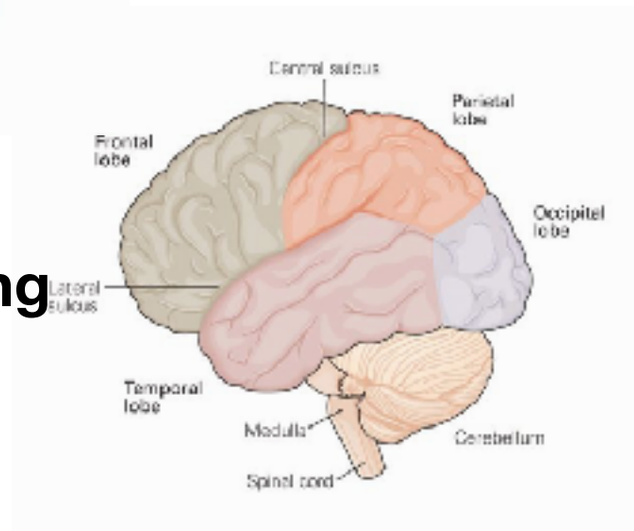


**Theoretical Insights** ⇄ **Biological Phenomenons**

# Thank you!