Optimal Streaming Approximations for all Boolean Max 2-CSPs and Max k-SAT

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Harvard University

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Motivation and Spoiler

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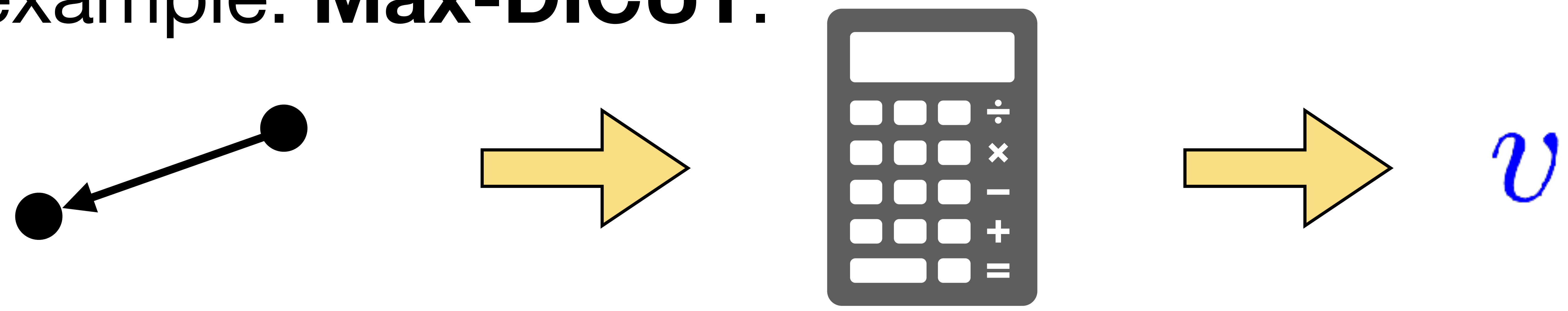

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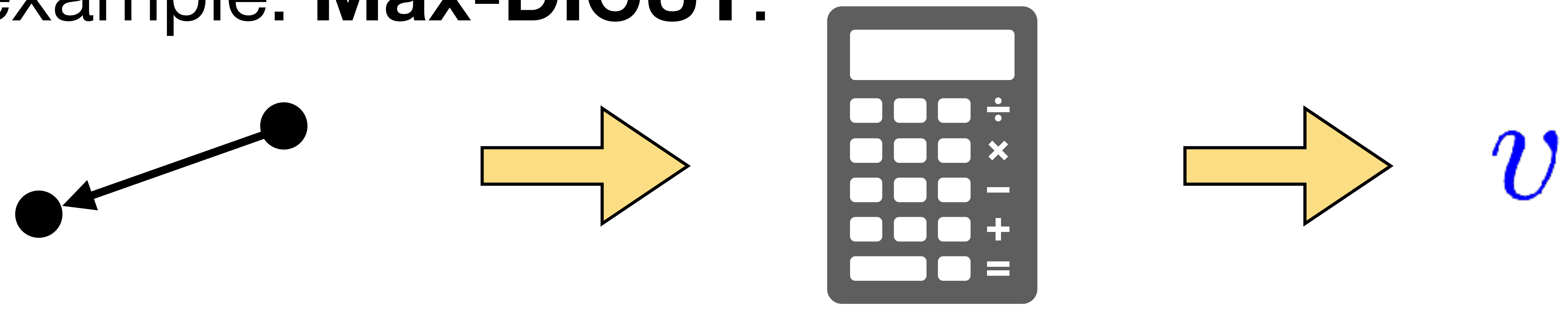

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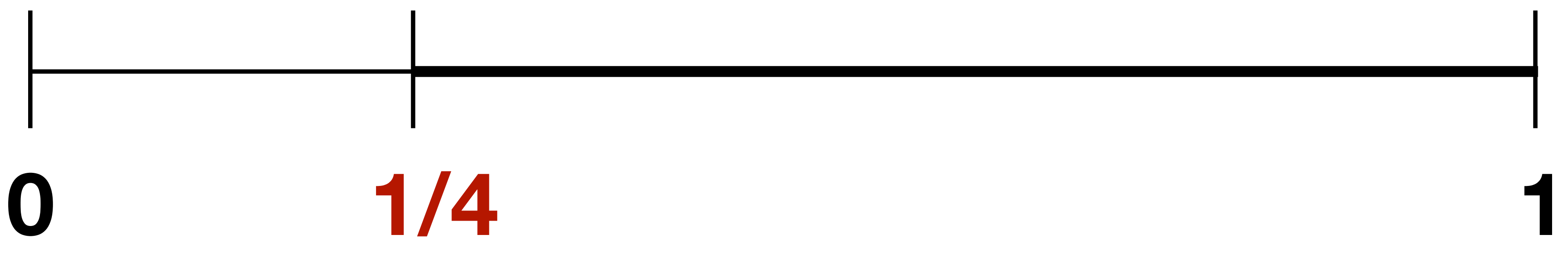

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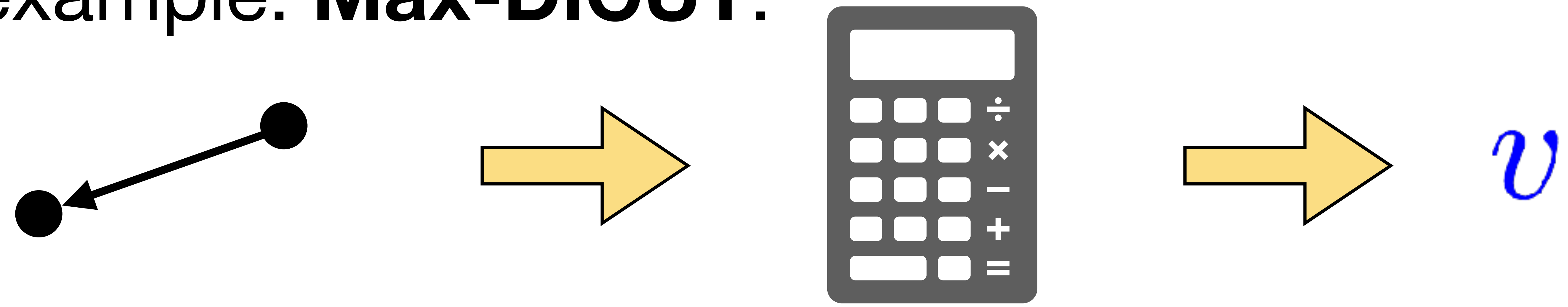

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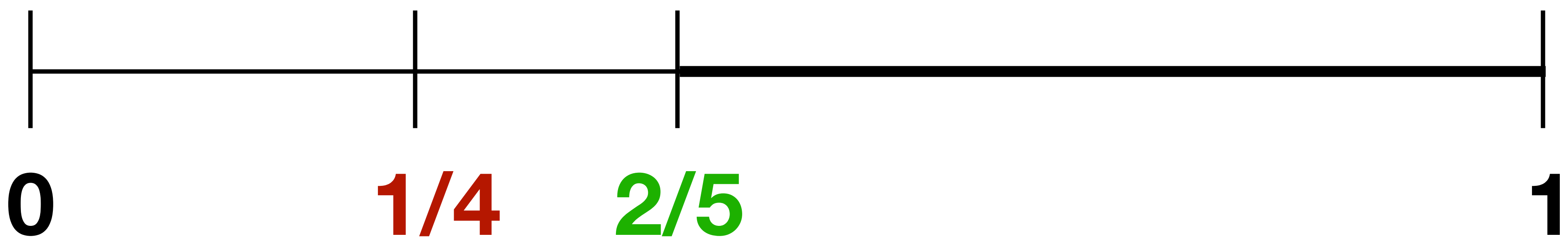

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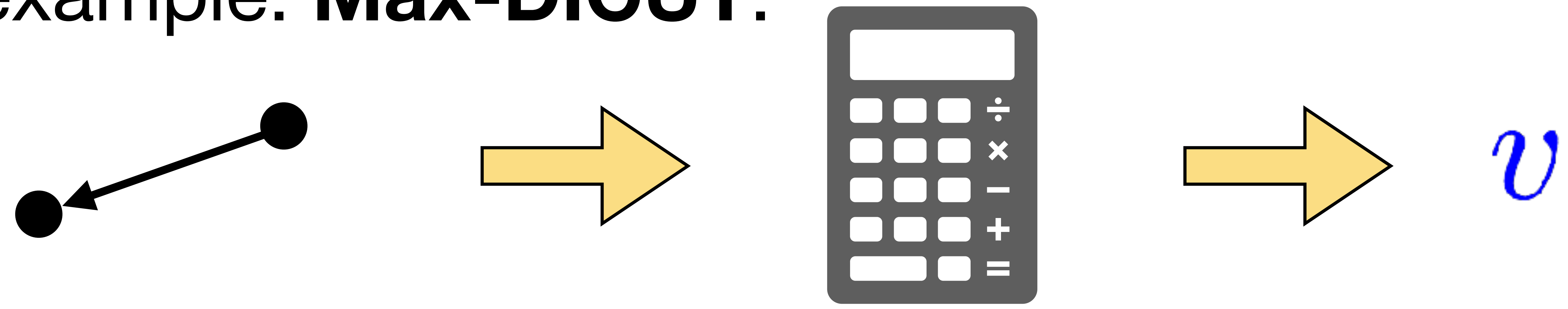

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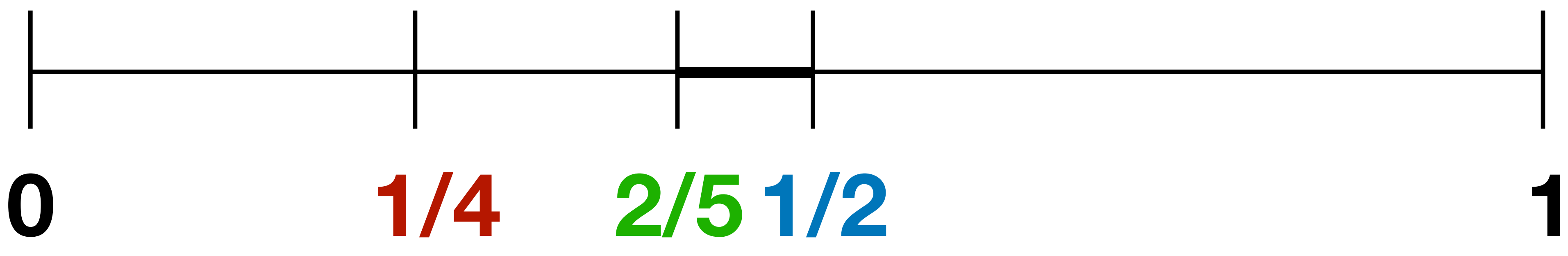

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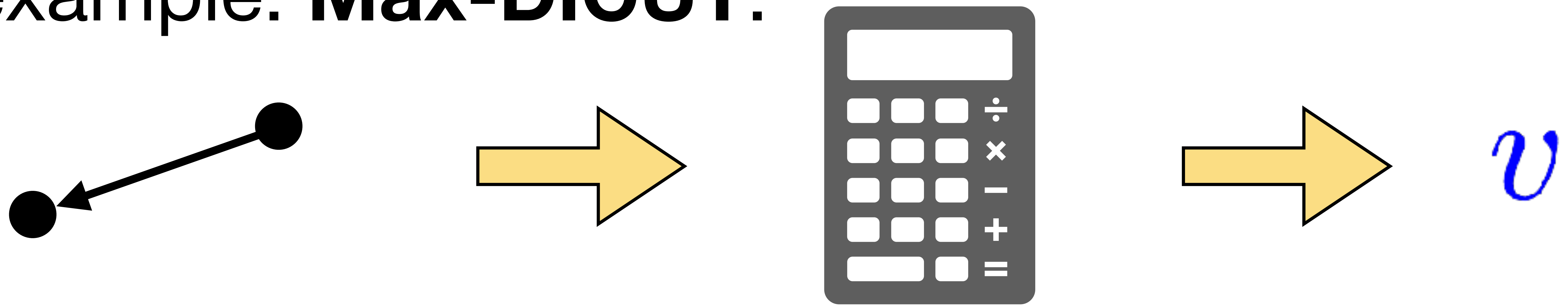

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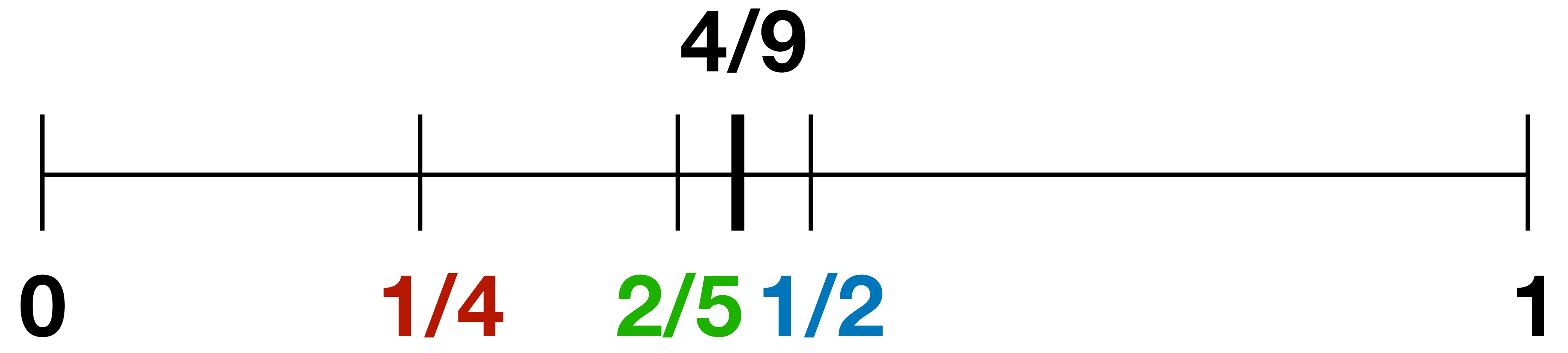

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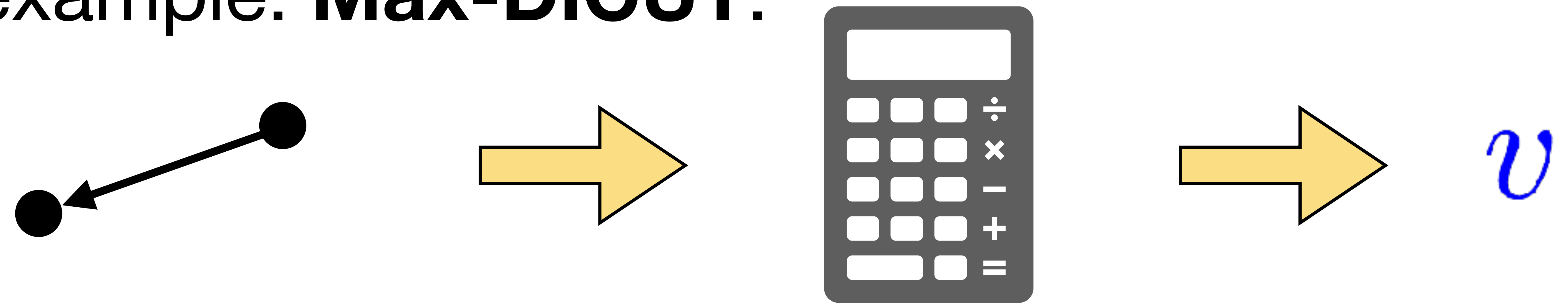

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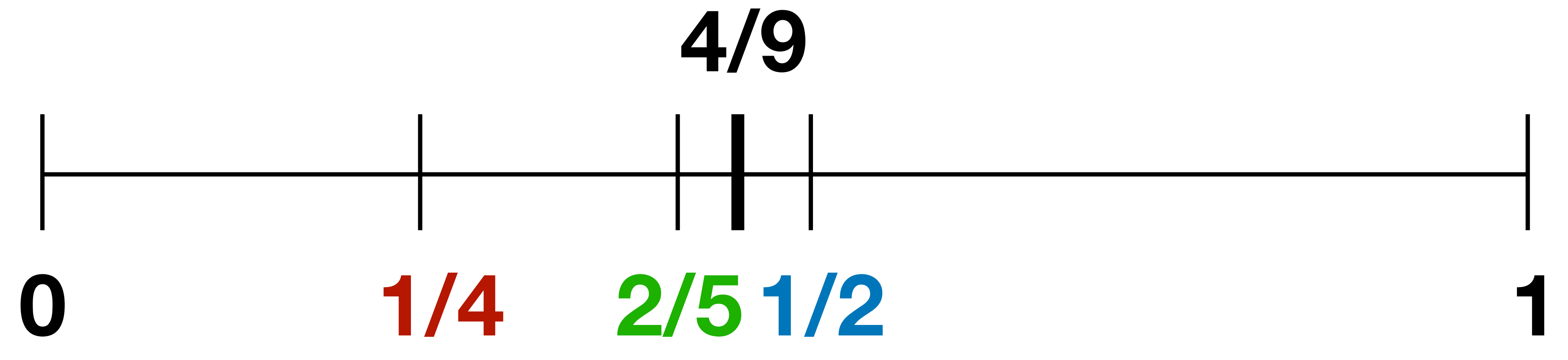

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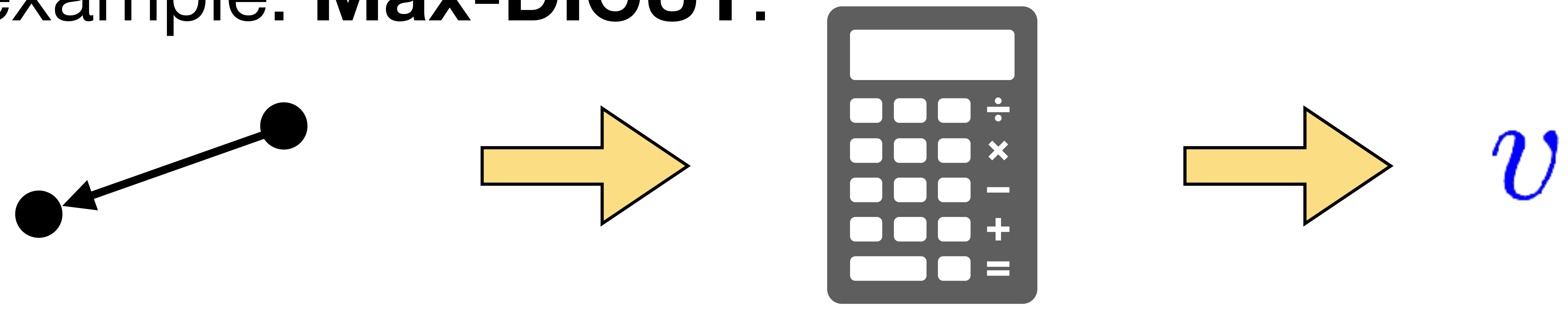

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- Further, we characterize the approximation ratio of every boolean 2-CSP!

Definitions

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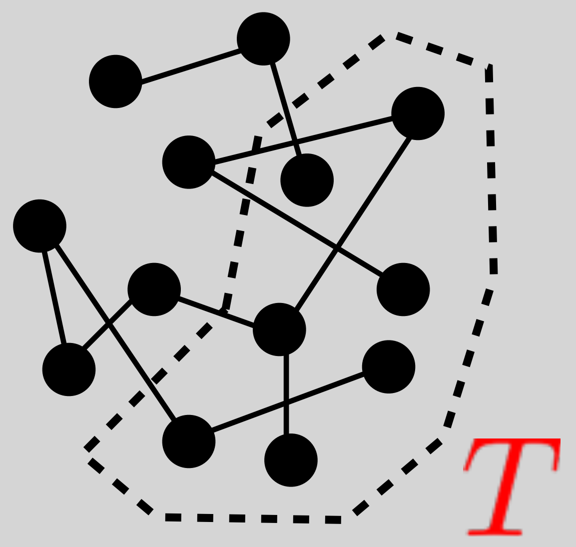

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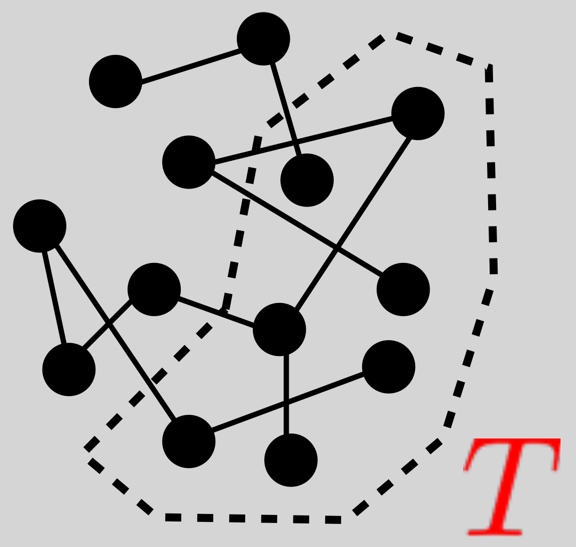

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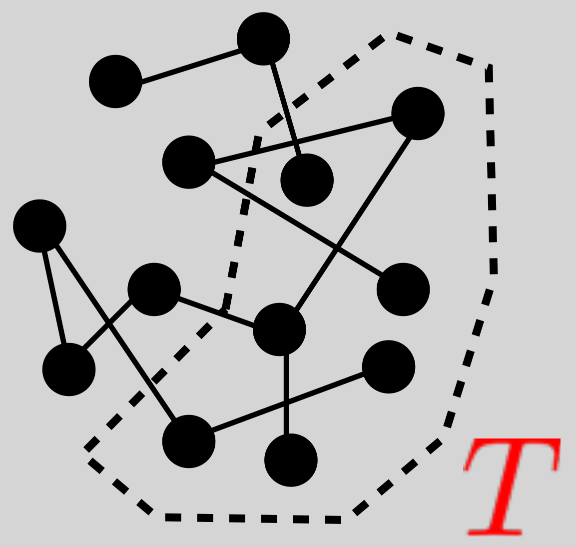

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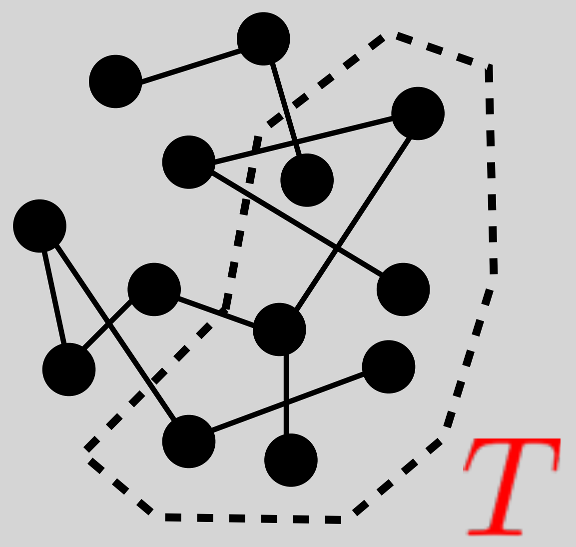

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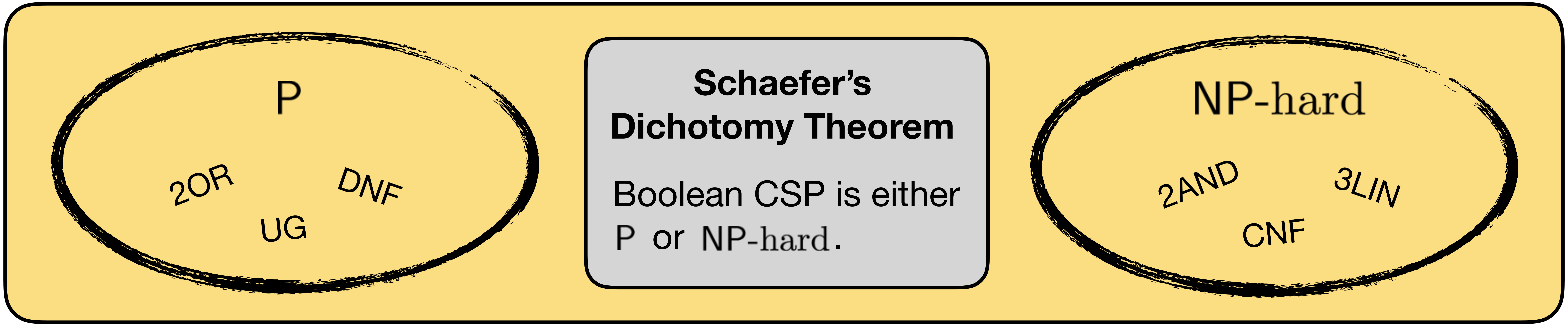

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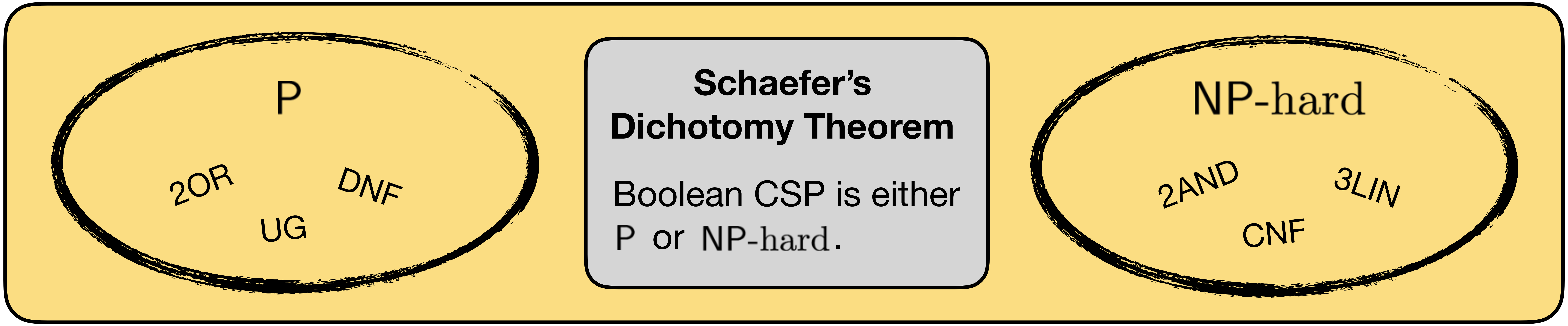

- What about solving CSP **approximately**?

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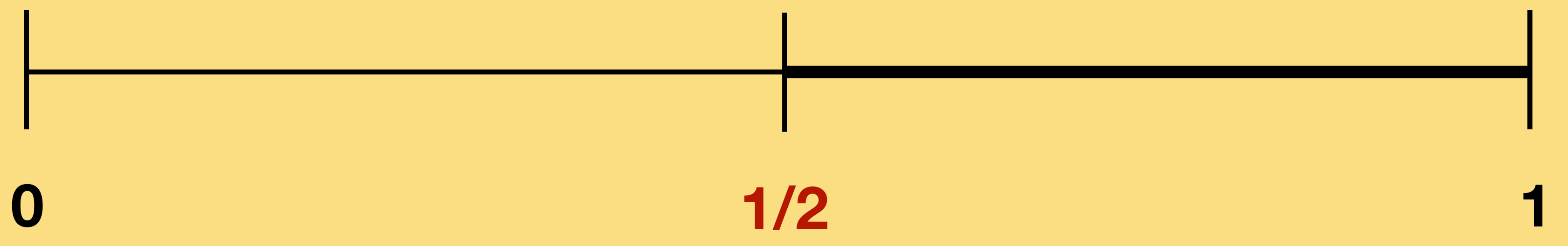

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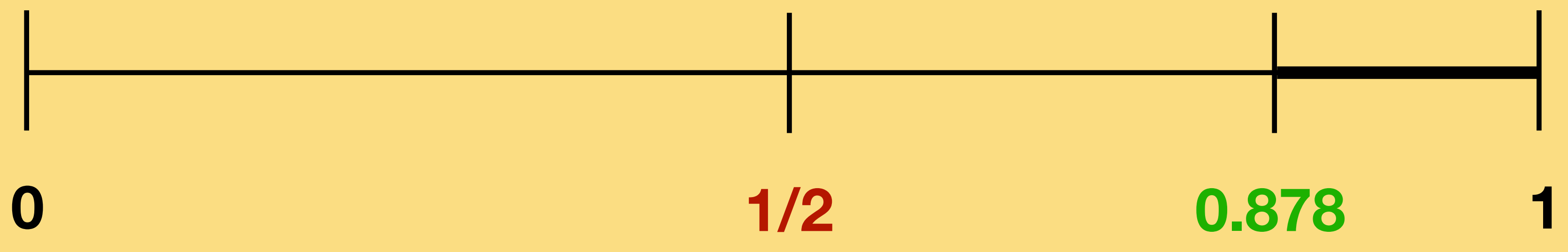

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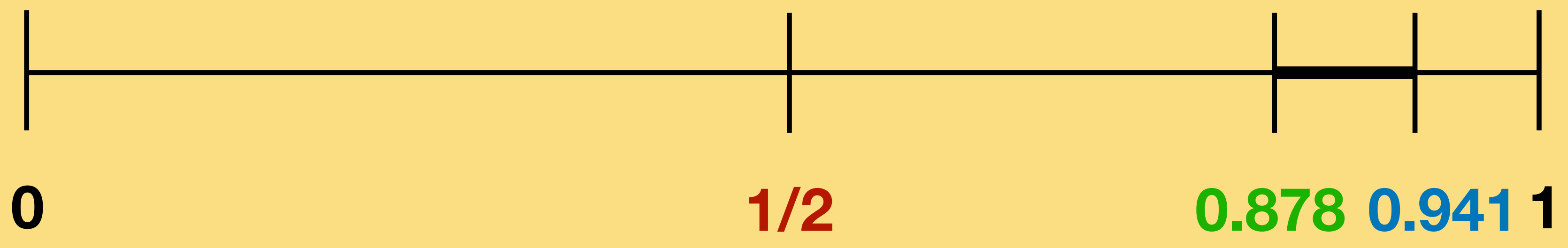

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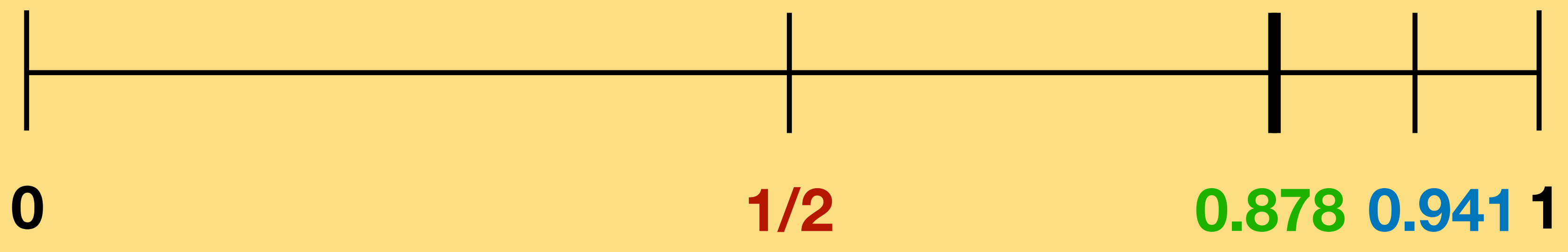

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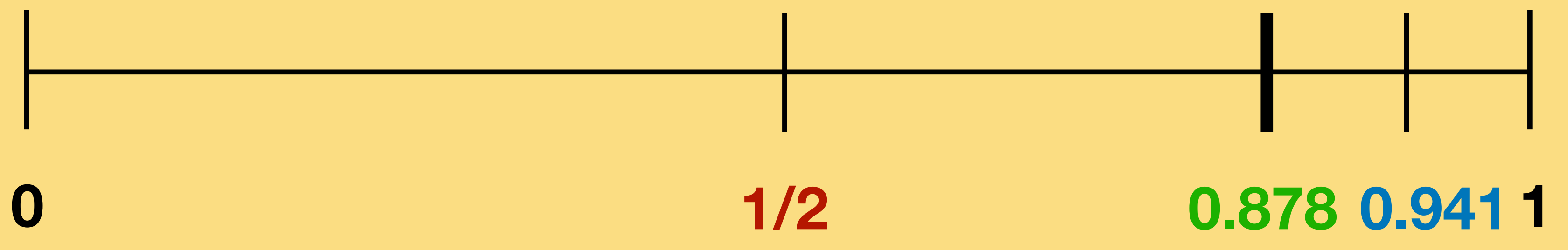

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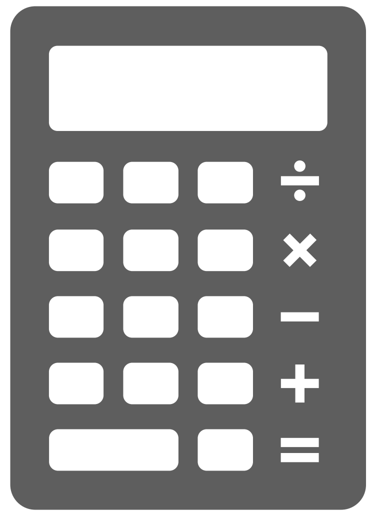

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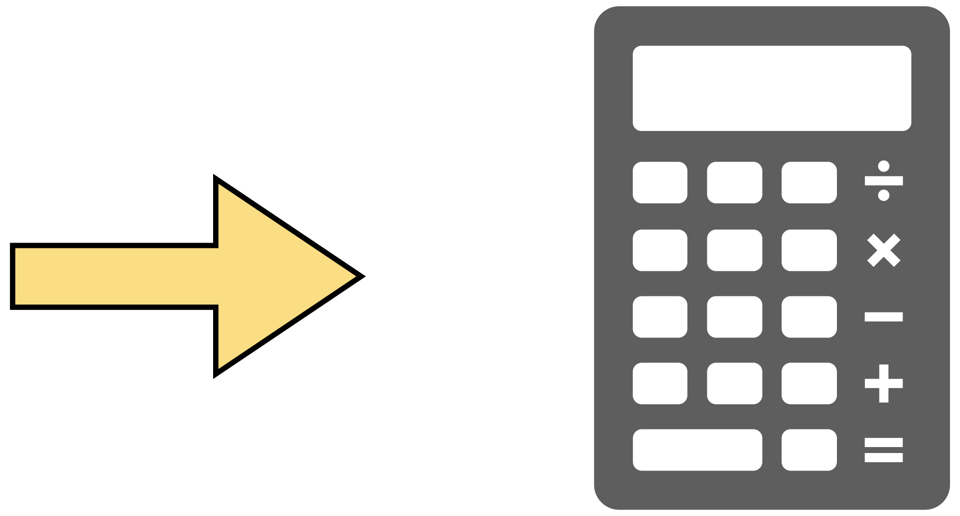

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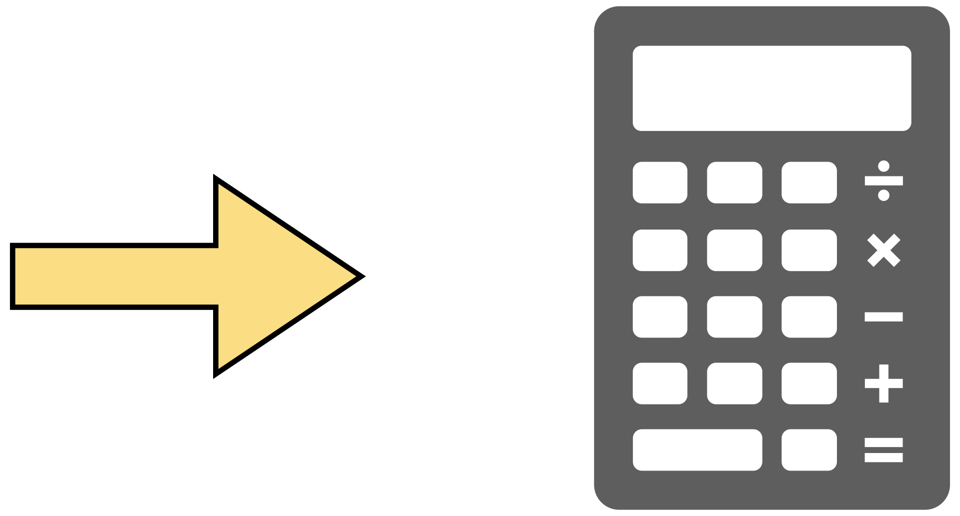
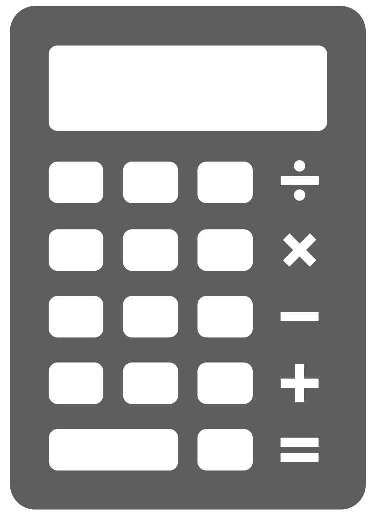
$$\neg x_5 \wedge x_2 \wedge \neg x_{23}$$



CSP in the Streaming Model

- The input (each constraint) arrives in a stream.

$$x_2 \wedge x_7 \wedge \neg x_{11}$$



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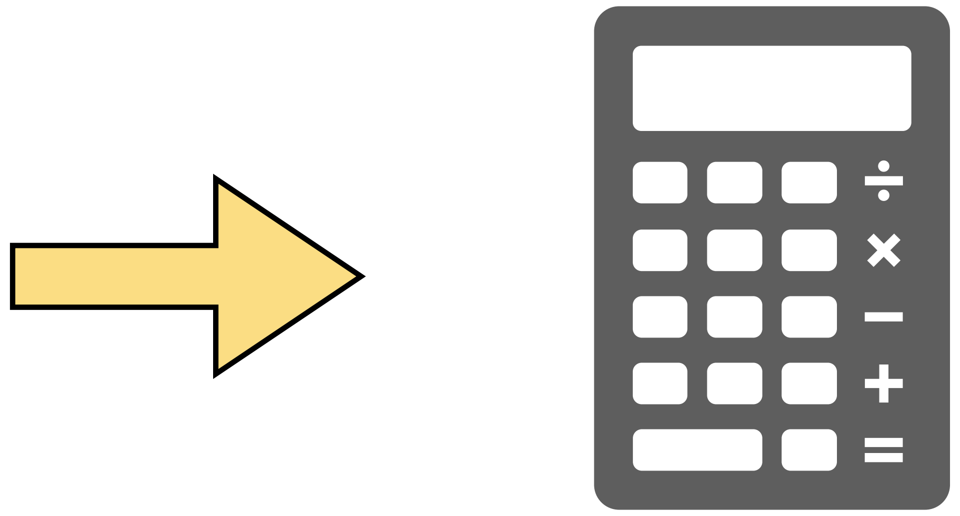
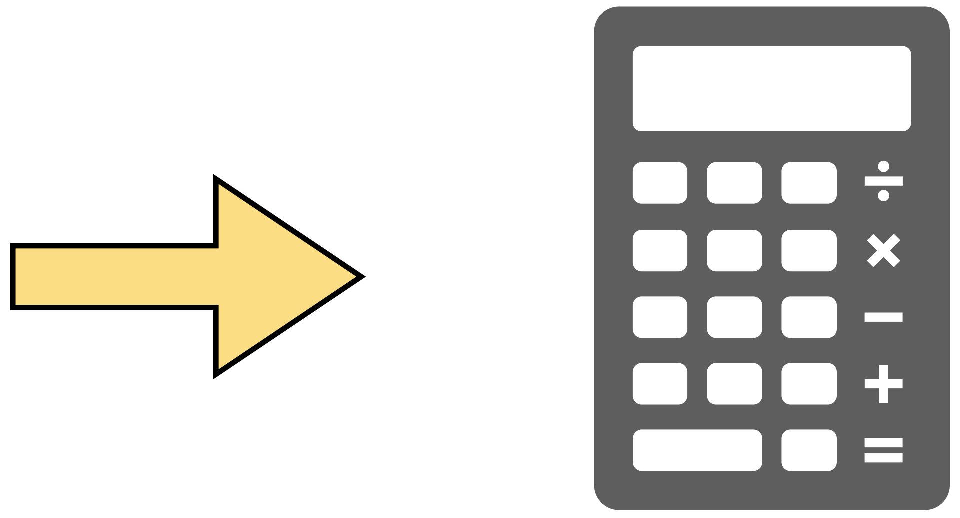
$$\neg x_7 \wedge x_{15} \wedge \neg x_{31}$$



CSP in the Streaming Model

- The input (each constraint) arrives in a stream.

$$\neg x_2 \wedge \neg x_8 \wedge \neg x_{12}$$



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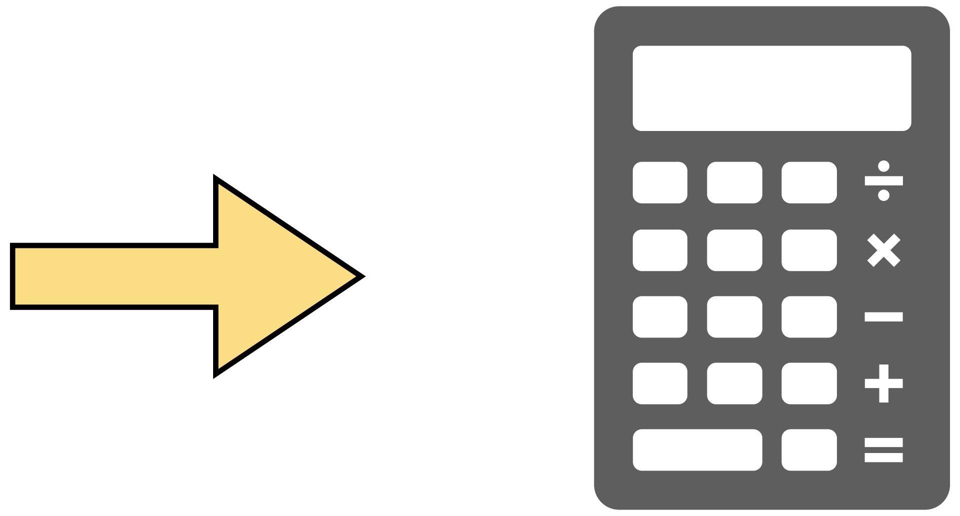

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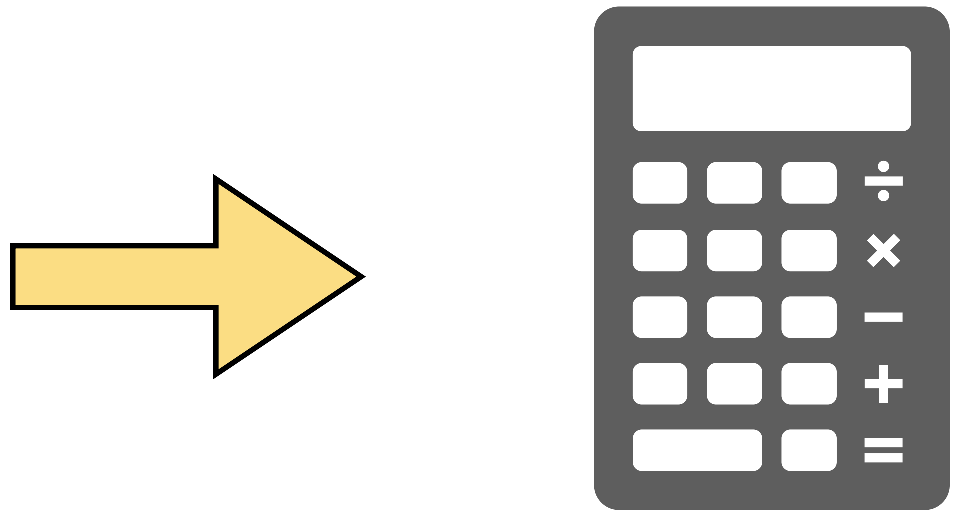

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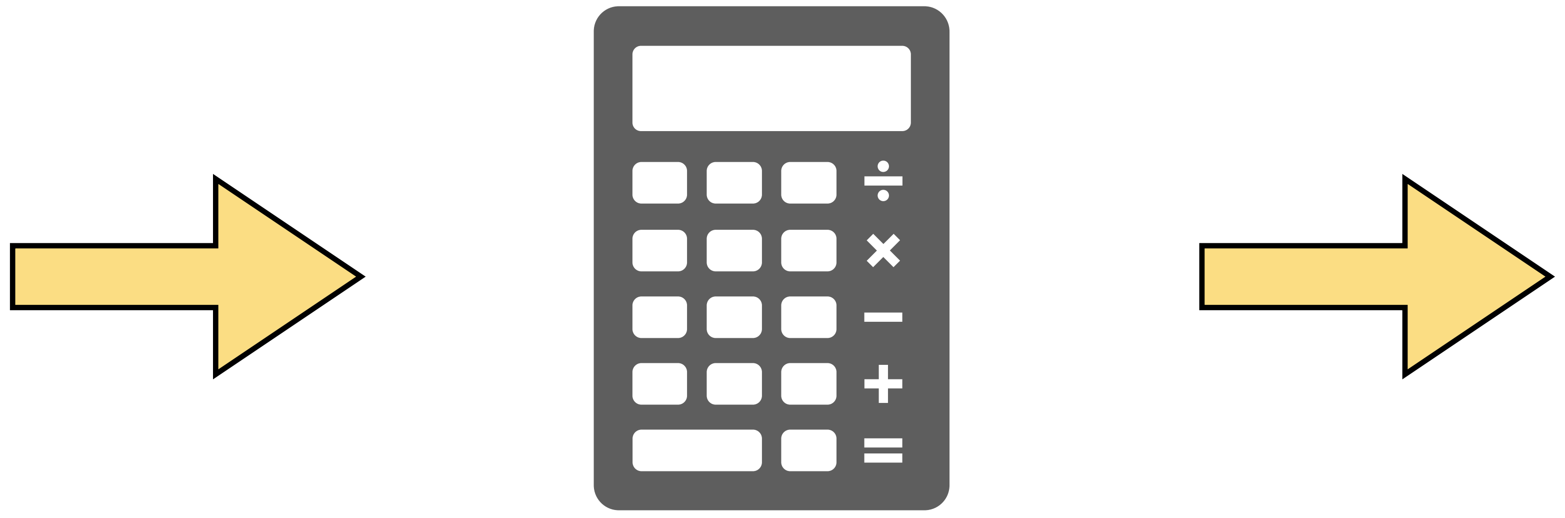

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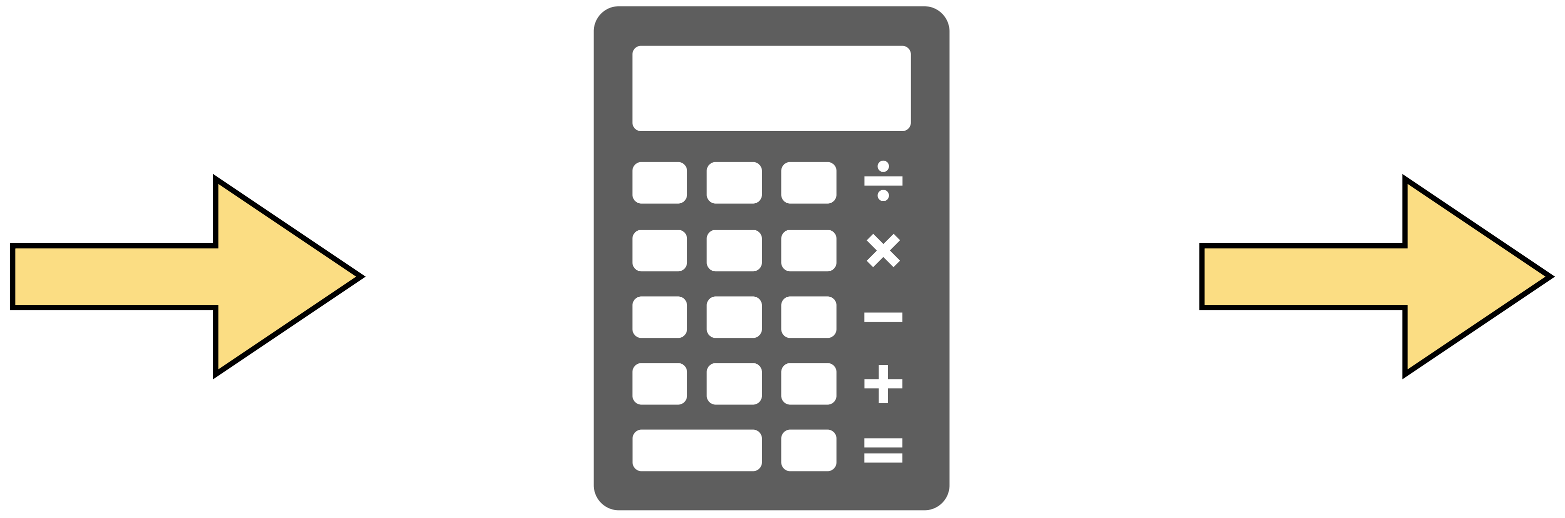

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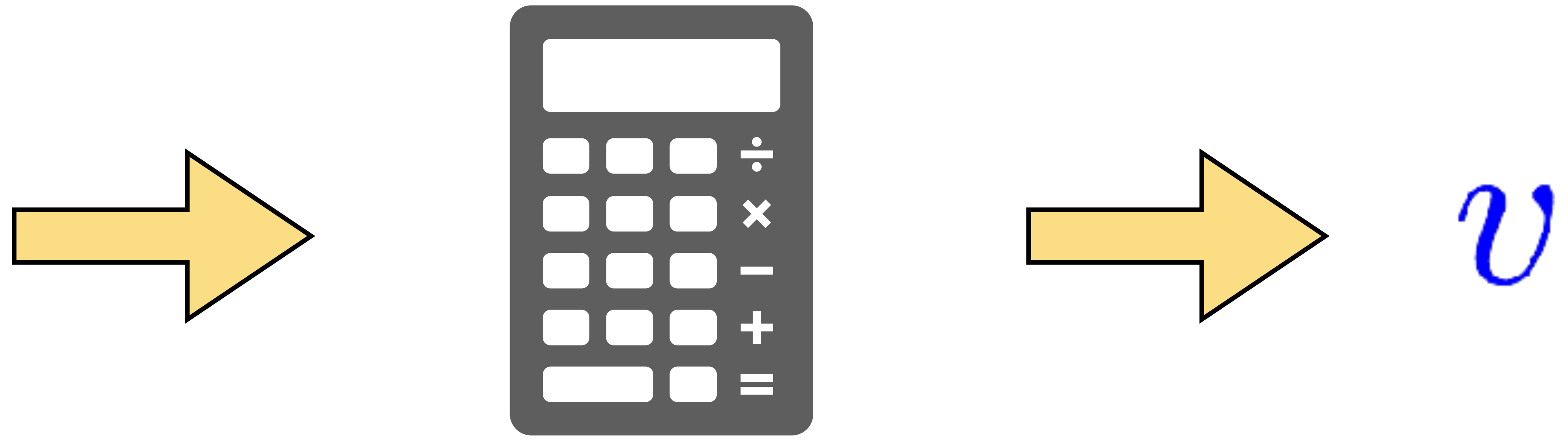

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Trivial Random Sampling is Optimal for Max-CUT!

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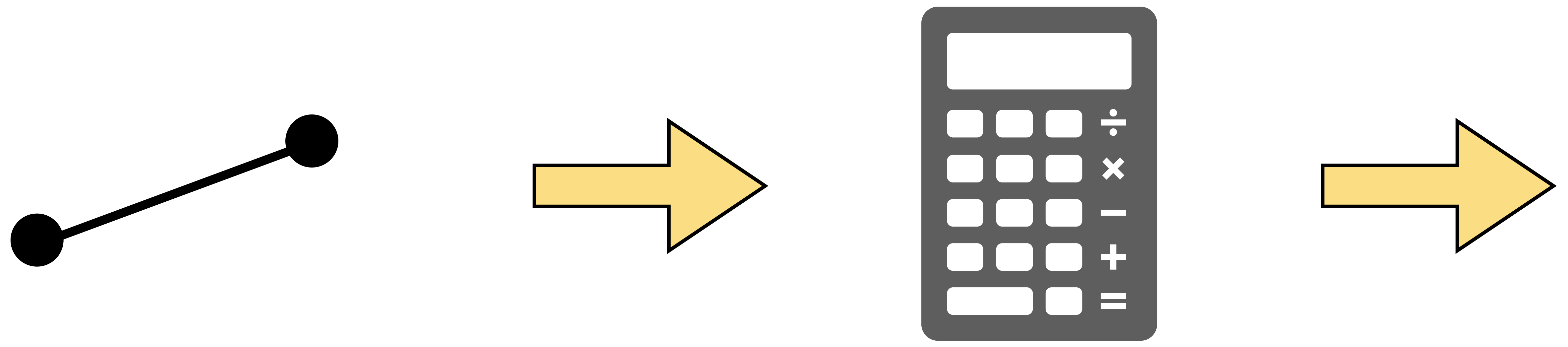
- Trivial random sampling gives **1/2**-approximation using $O(\log n)$ space.



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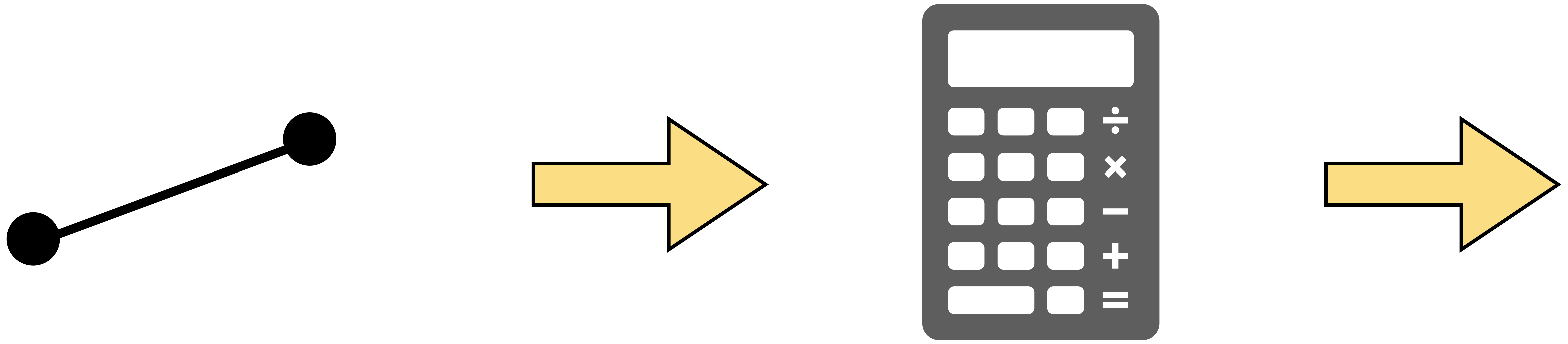

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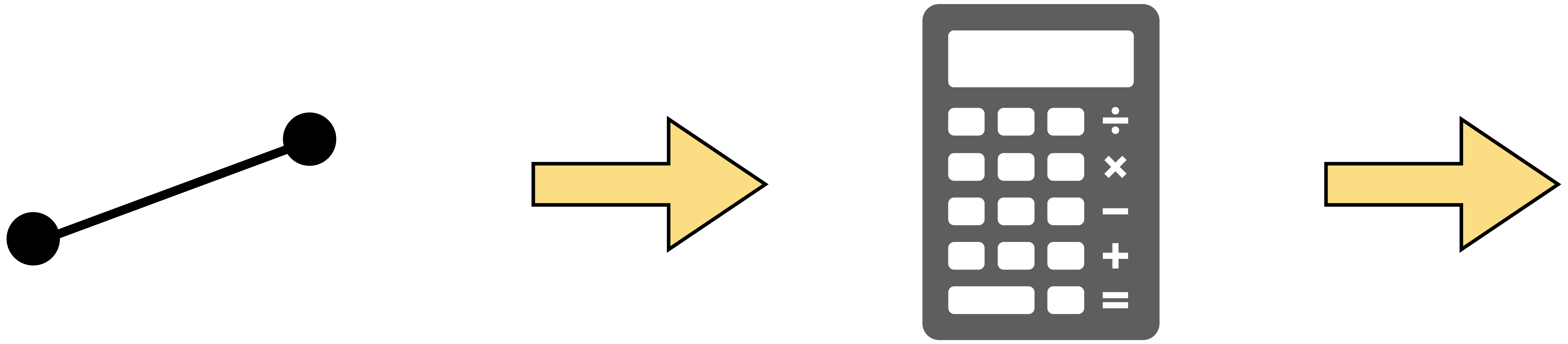

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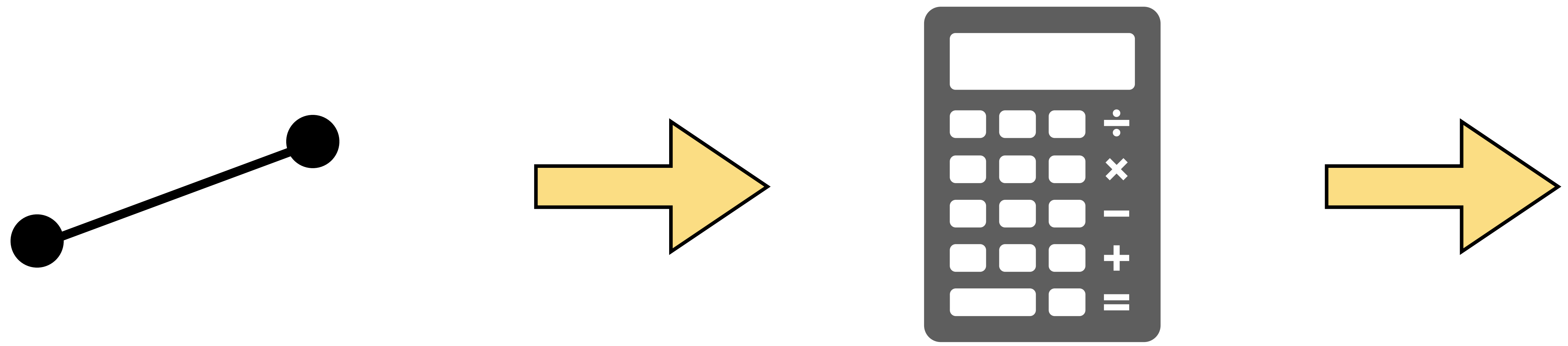

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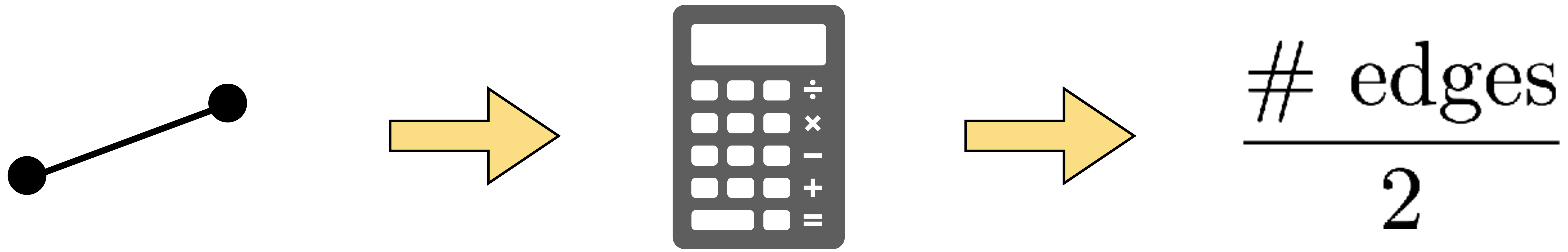

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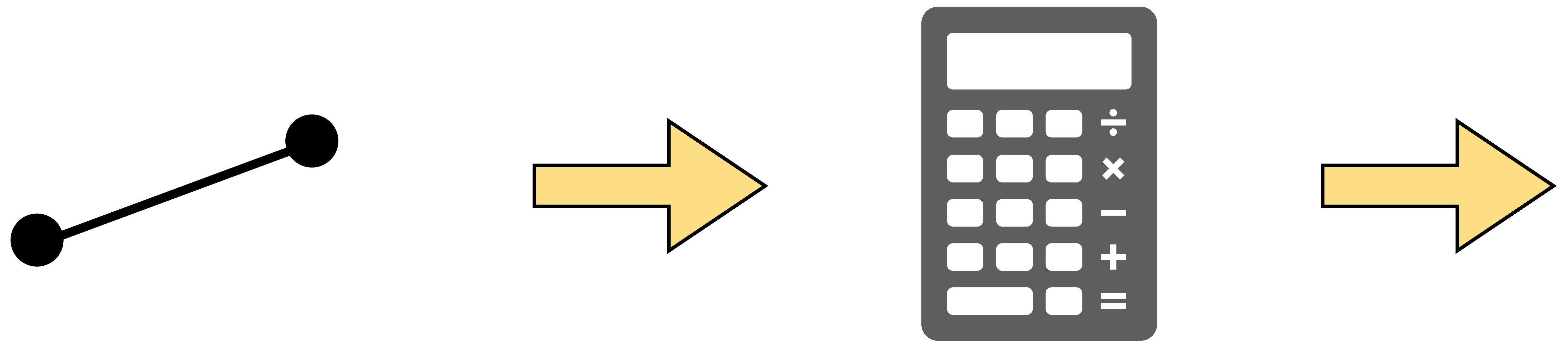

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There's a SDP-based algorithm which gives **0.878**-approx.

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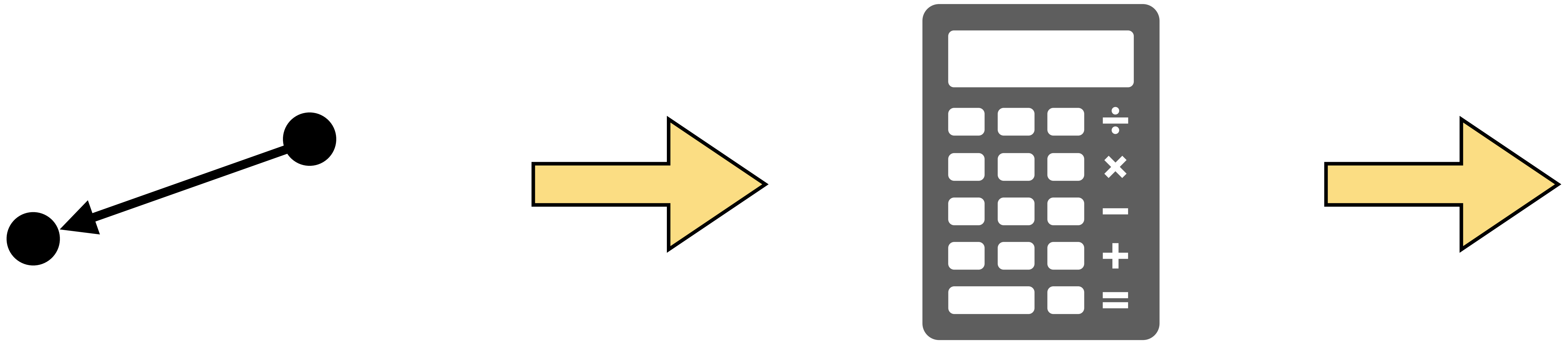

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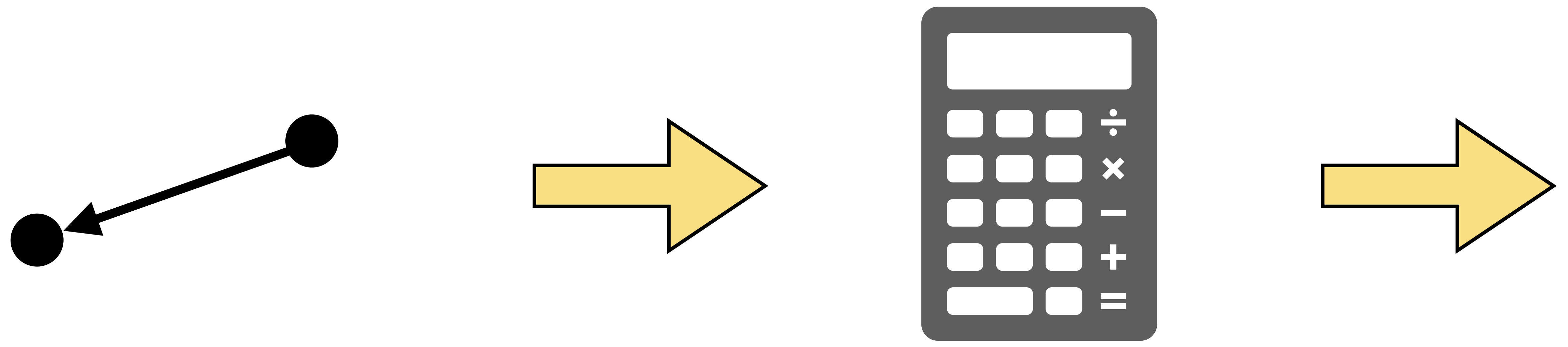

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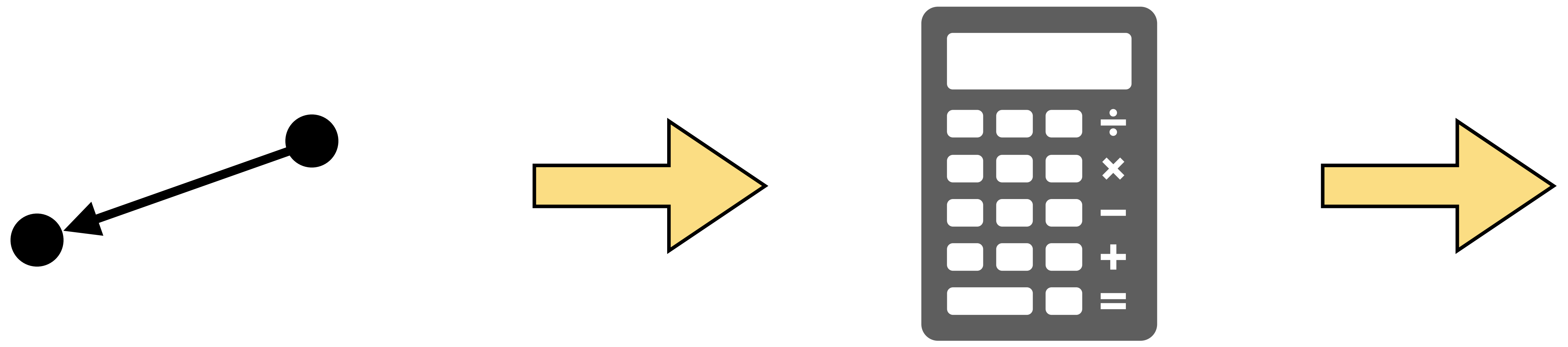

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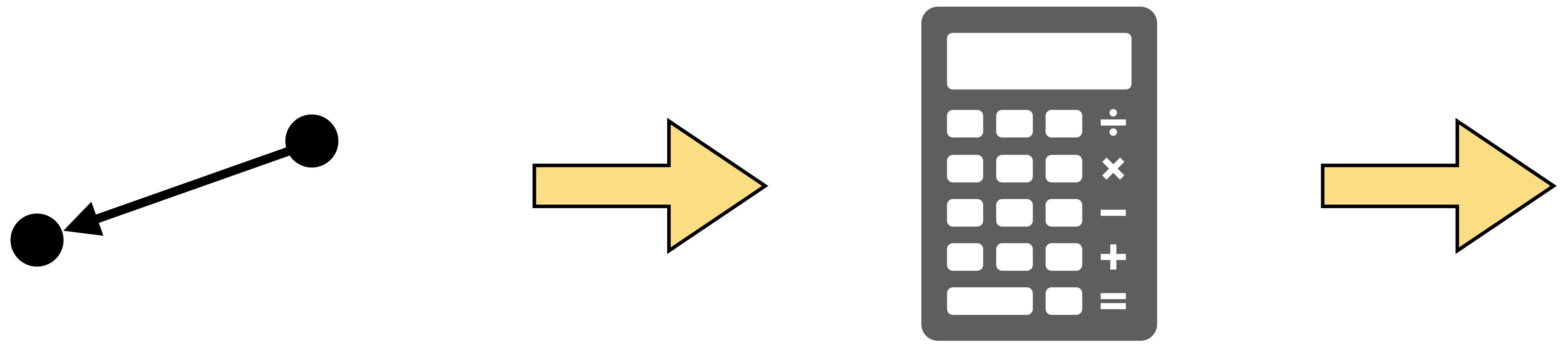

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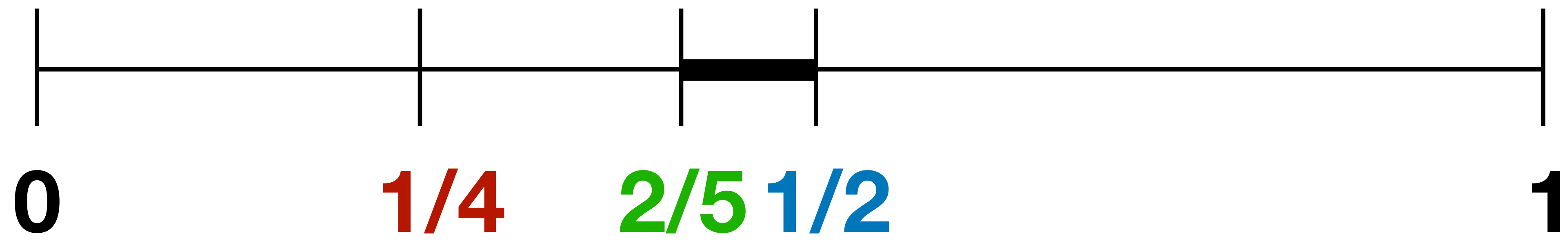

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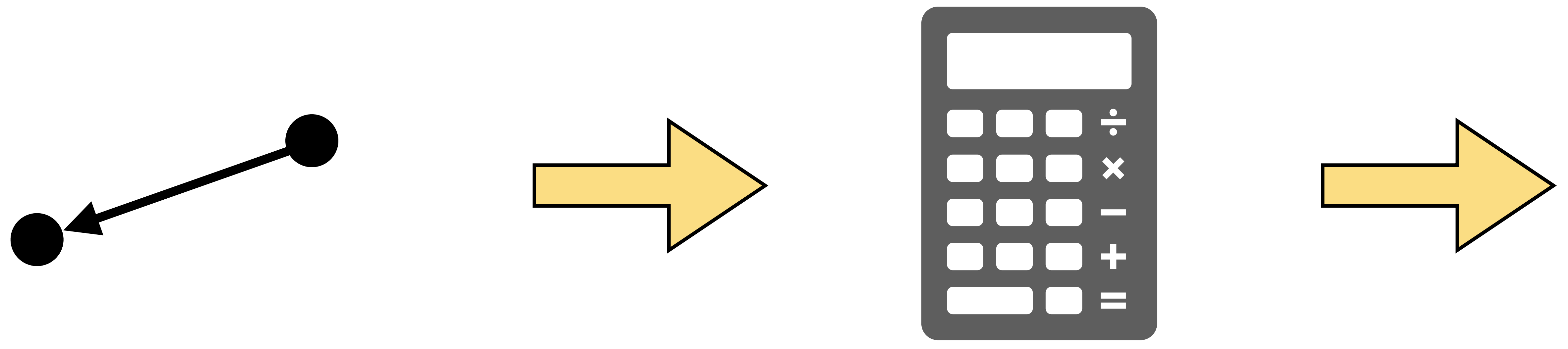

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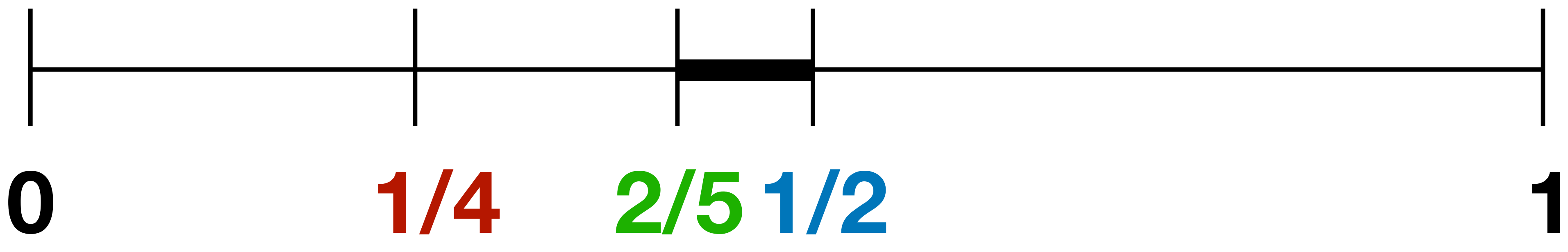
- What's the "right approximation ratio"?



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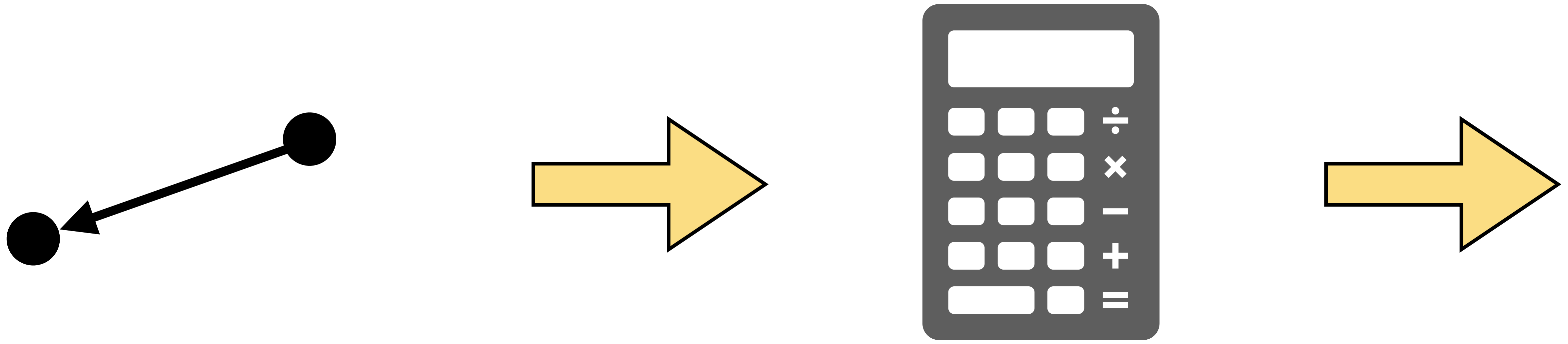

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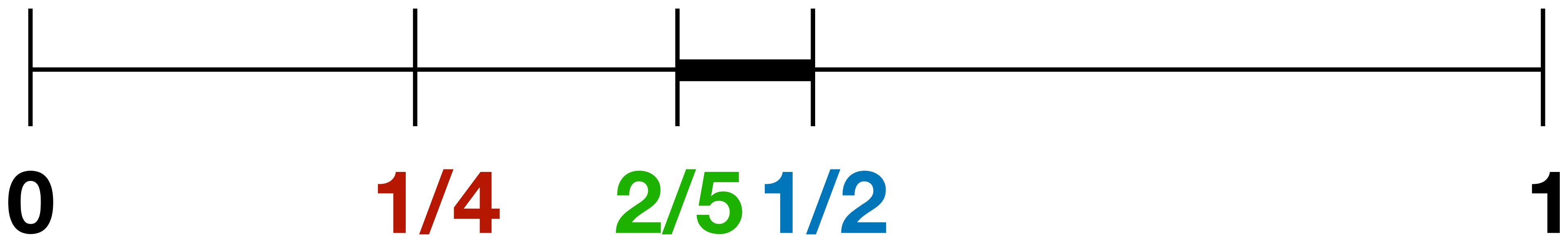
- What's the "right approximation ratio"?
- What about other CSP?



Our Results

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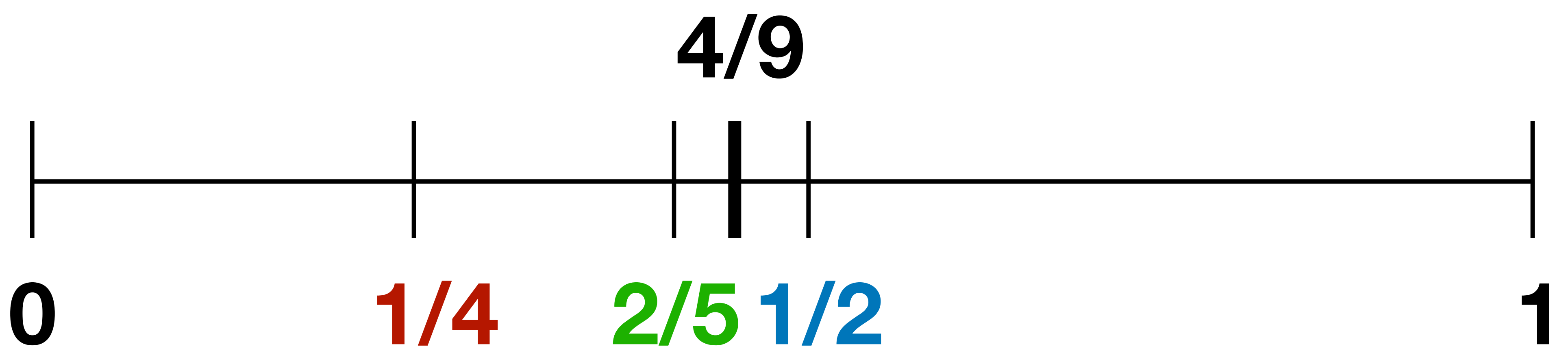
- The answer of Max-DICUT is **4/9** 😳



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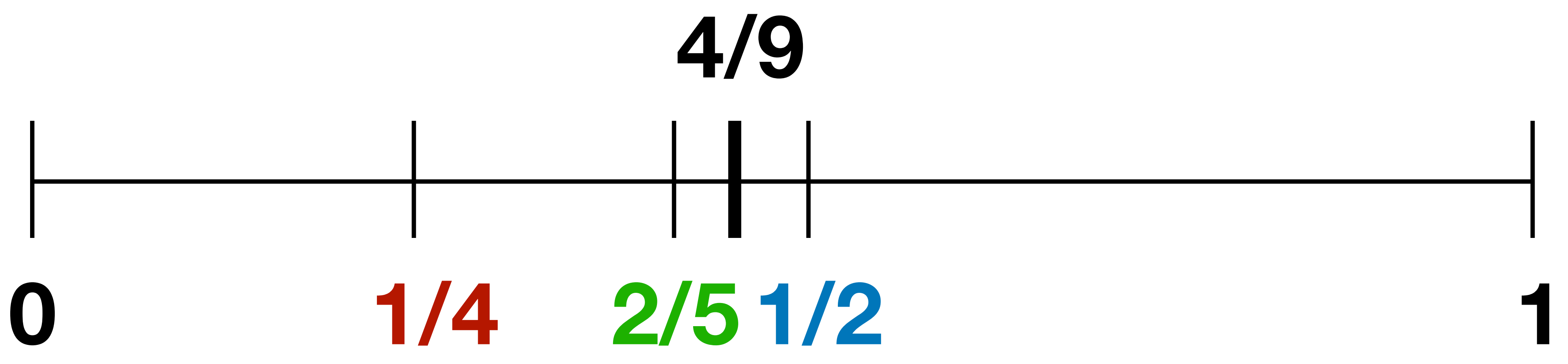

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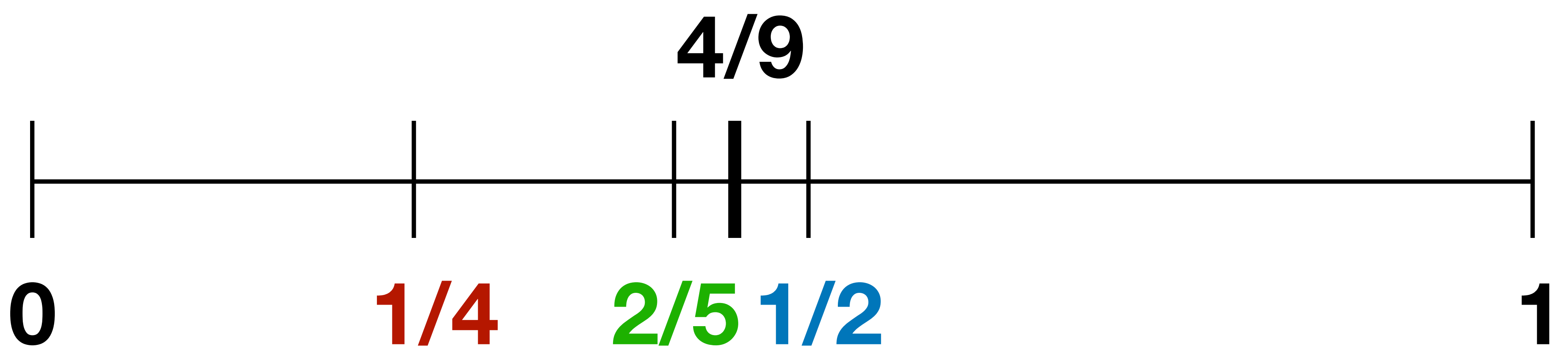

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Theorem (main).

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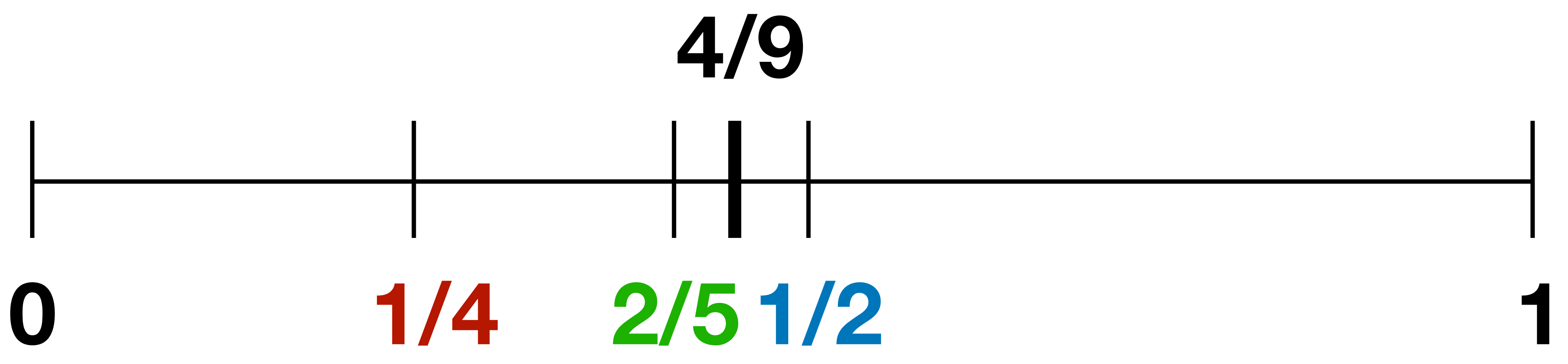

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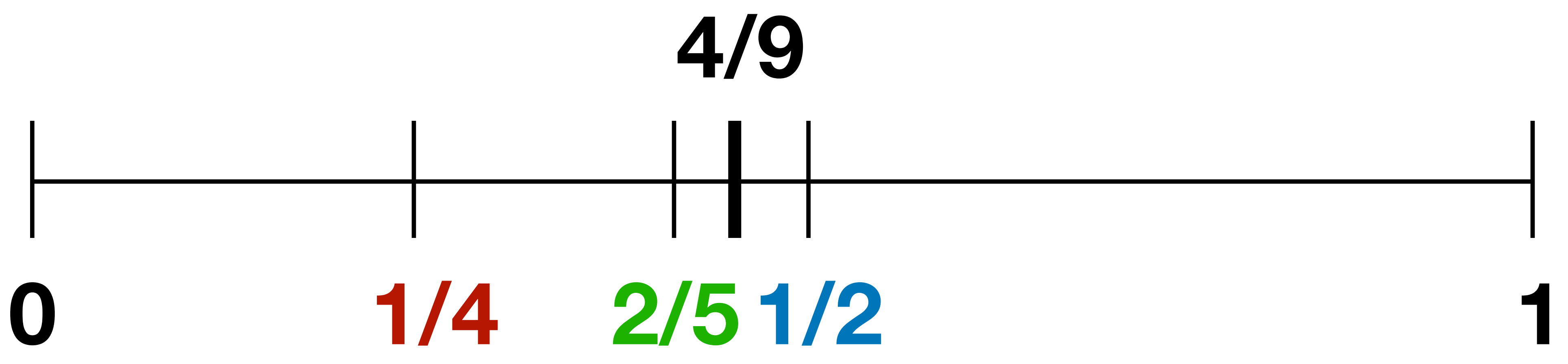

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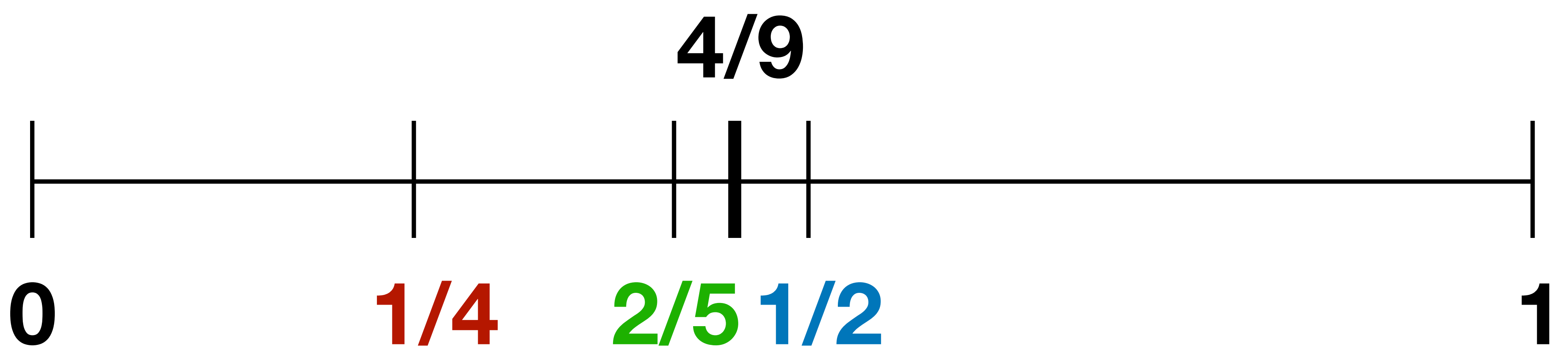

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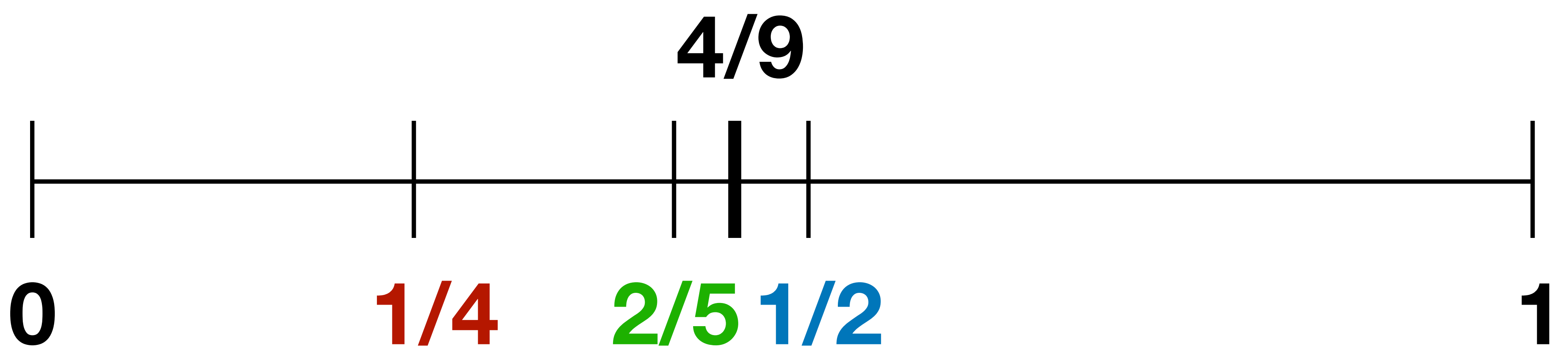

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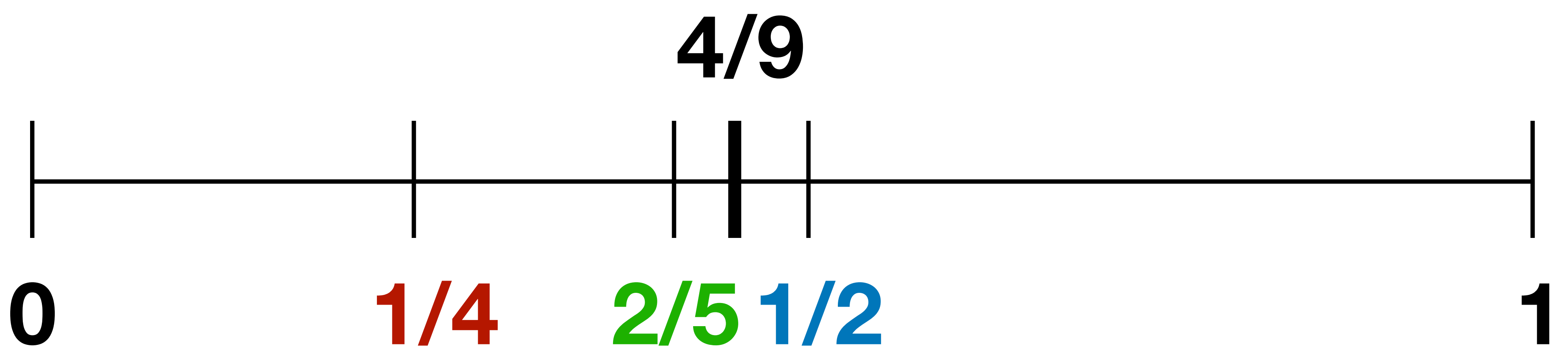

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Λ	α_Λ	τ_Λ	Reference
XOR	$\frac{1}{2}$	1	[KK19]

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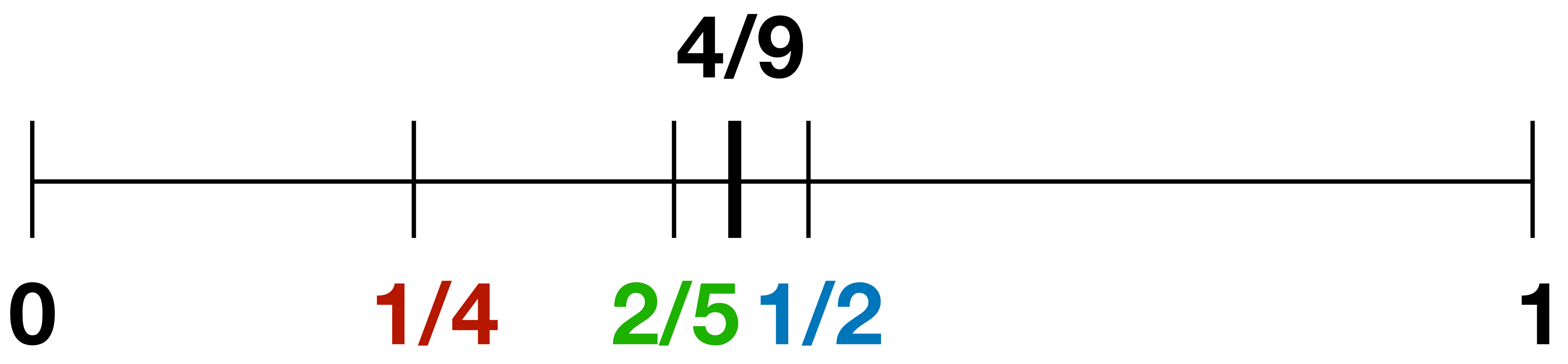

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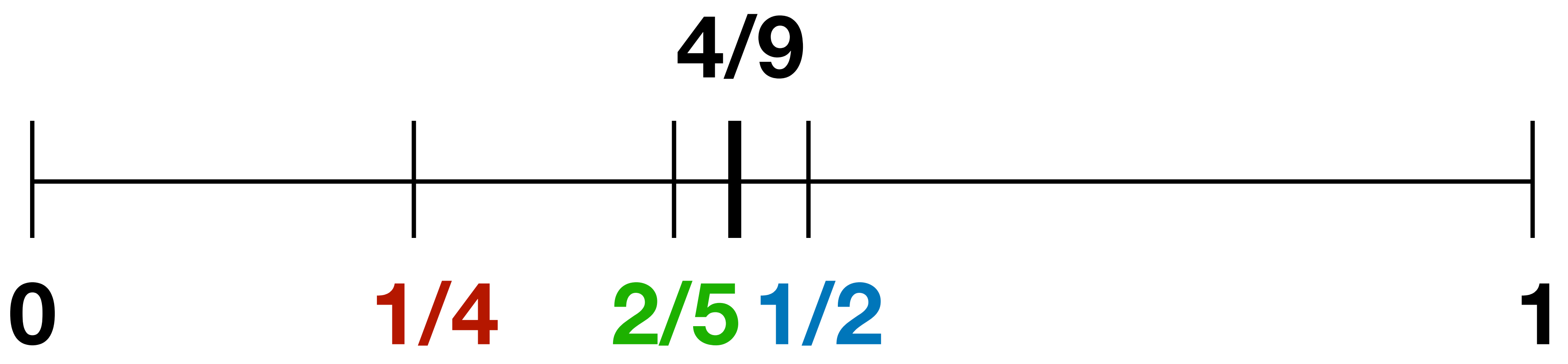

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EOR	$\frac{3}{4}$	$\frac{1}{2}$	This work

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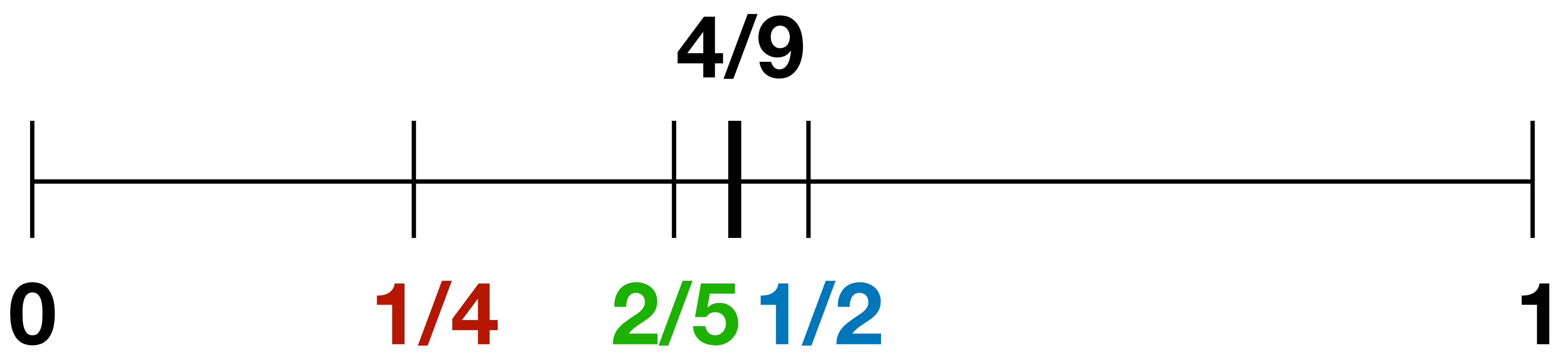

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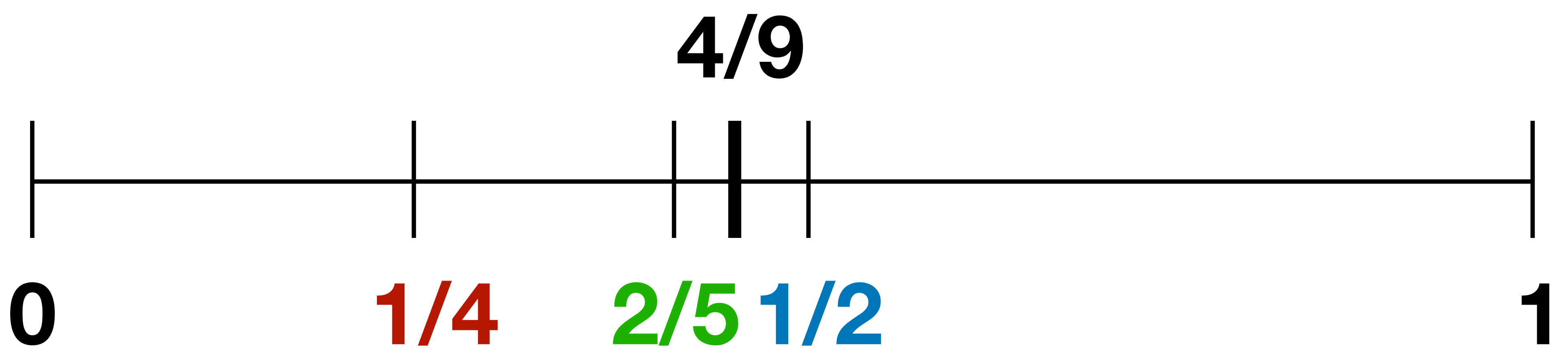

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EOR	$\frac{3}{4}$	$\frac{1}{2}$	This work
OR	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	This work

Can be extended to Max k-SAT!

Algorithms

Algorithms

with a focus on Max-DICUT

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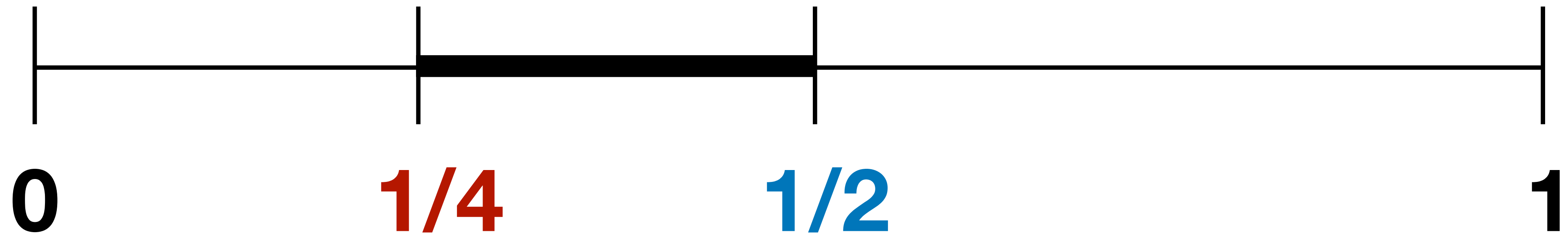

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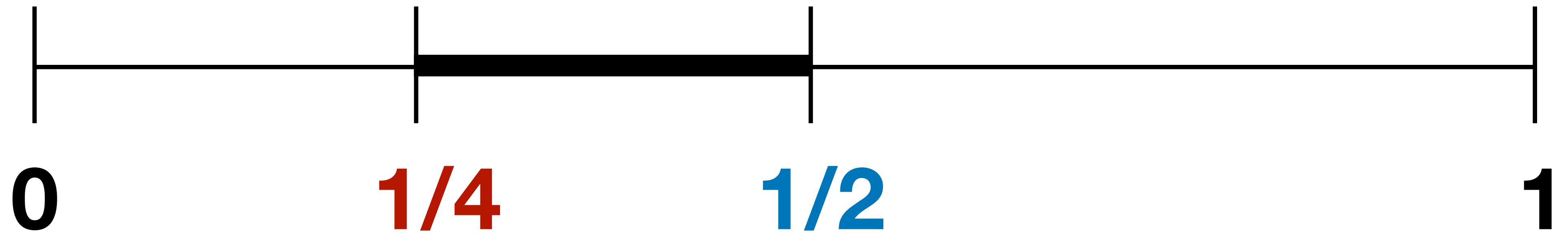

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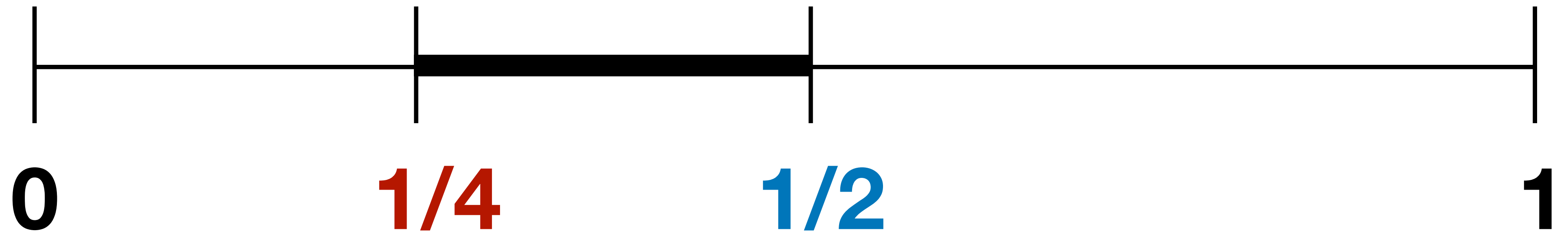

Definition (bias and total bias):

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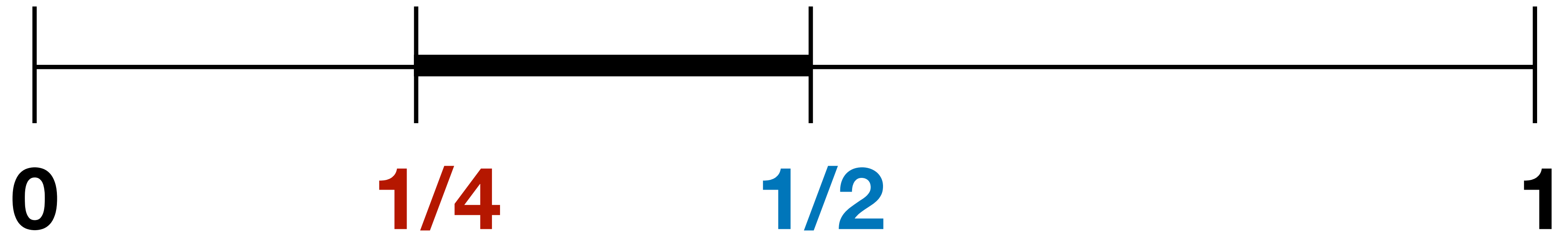

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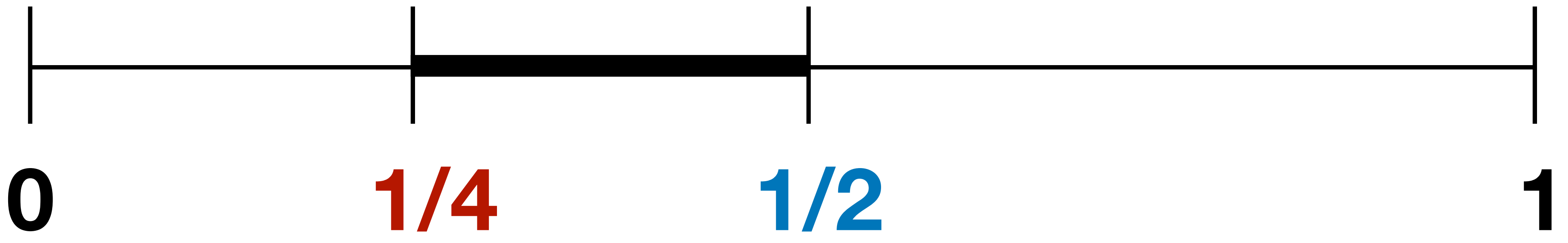

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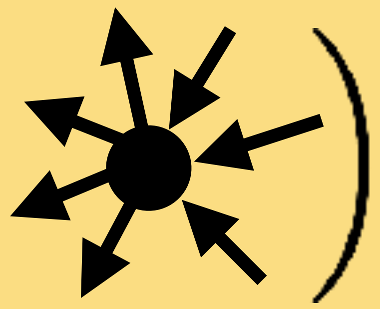
$\text{bias}\Big($  $\Big) = -1$

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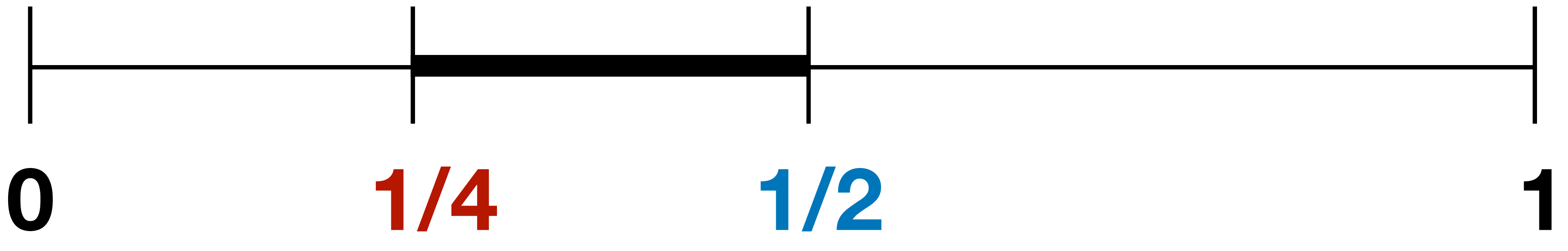

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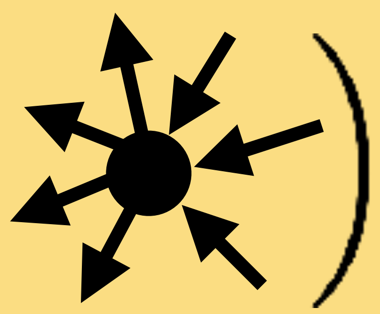
bias() $= -1$

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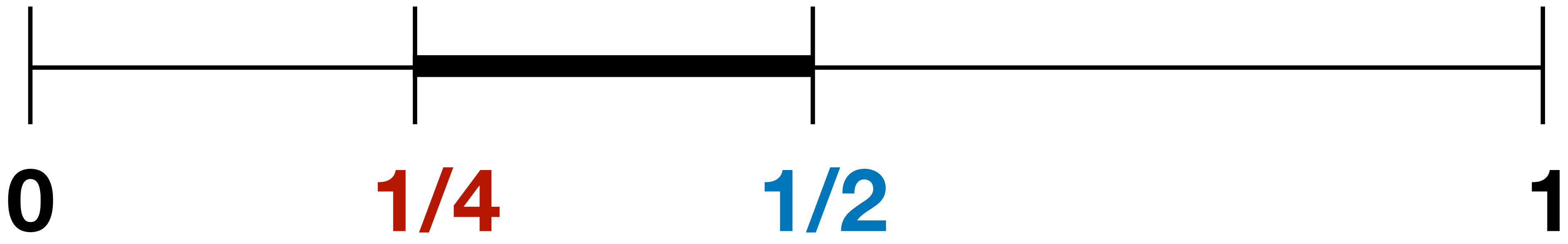

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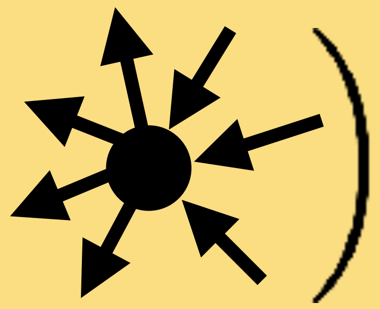
$\text{bias}\Big($  $\Big) = -1$

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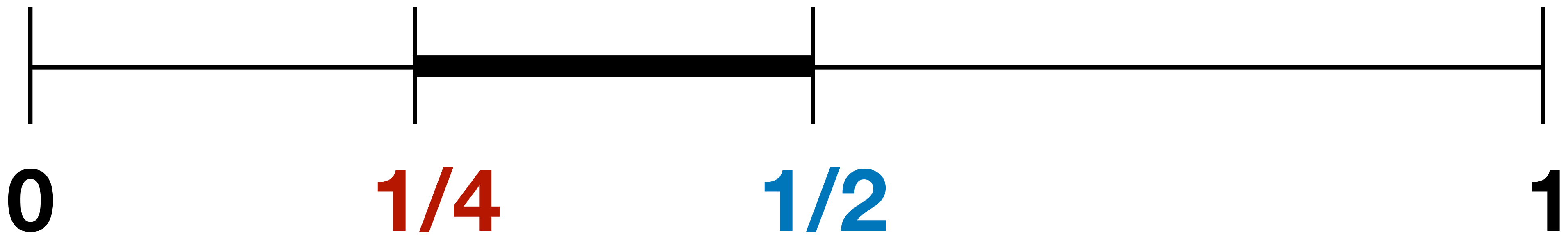

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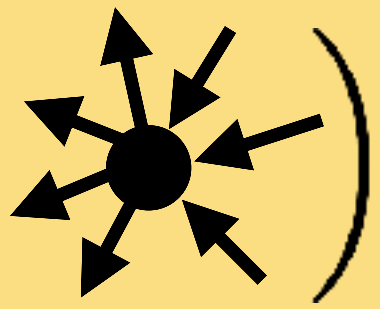
$\text{bias}($  $) = -1$

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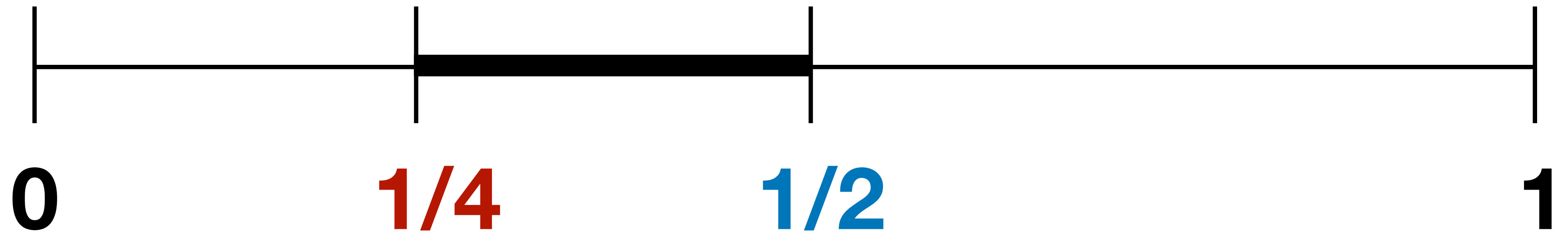

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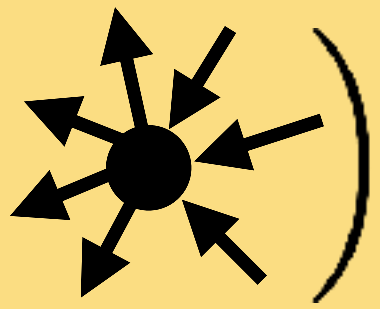
bias() $= -1$

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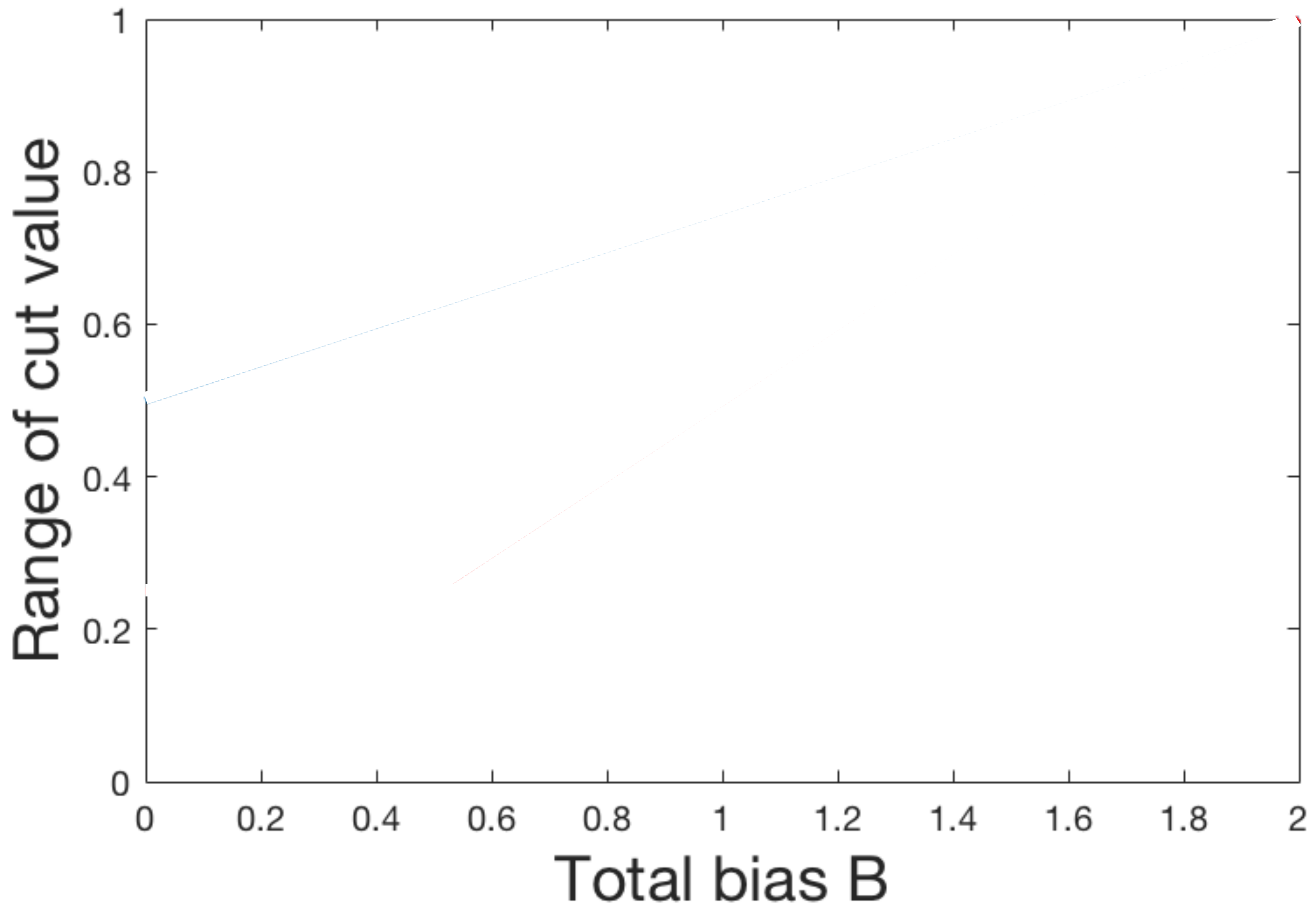

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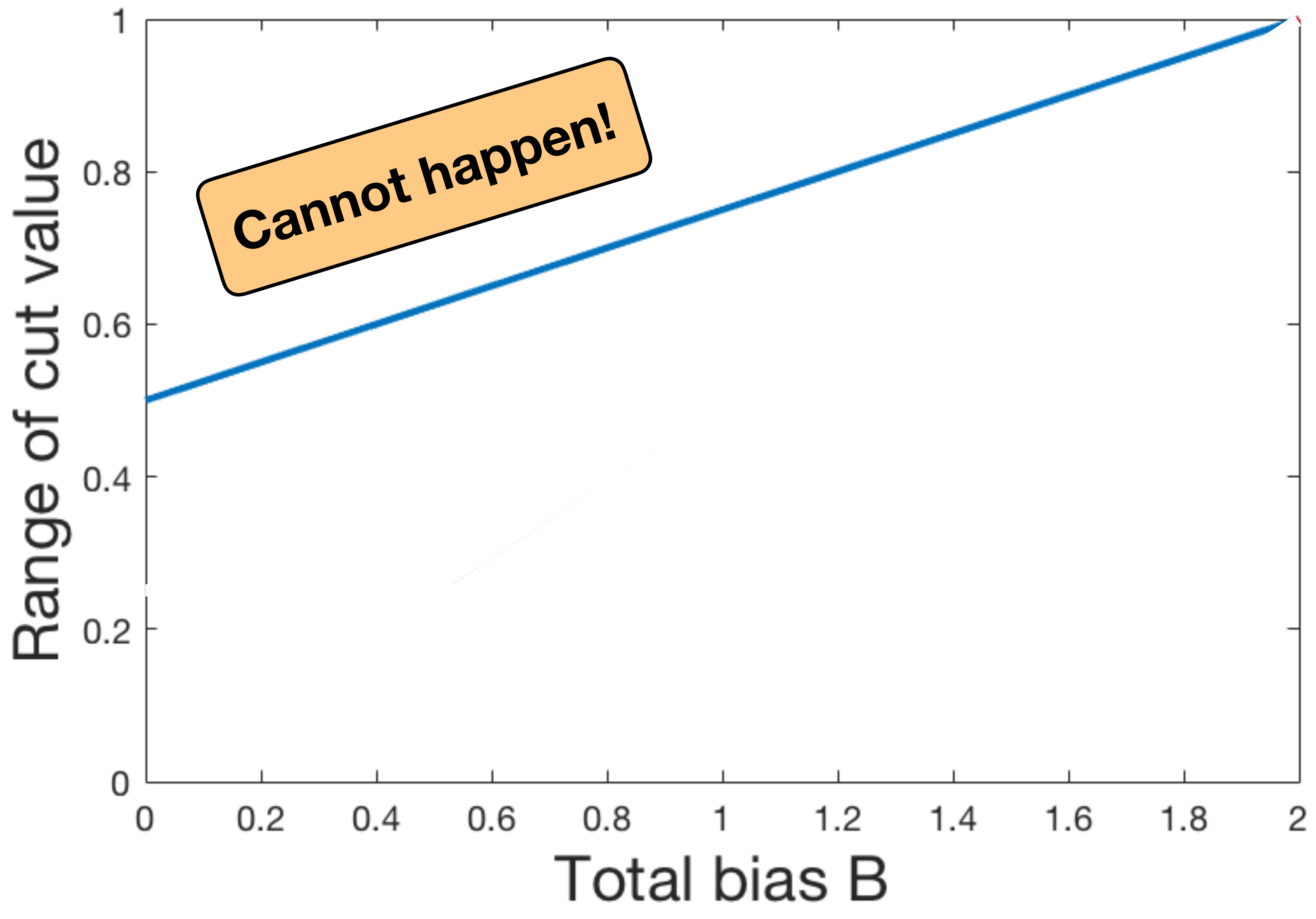

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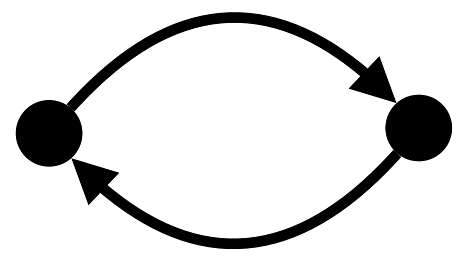
  Small bias => ∃ 

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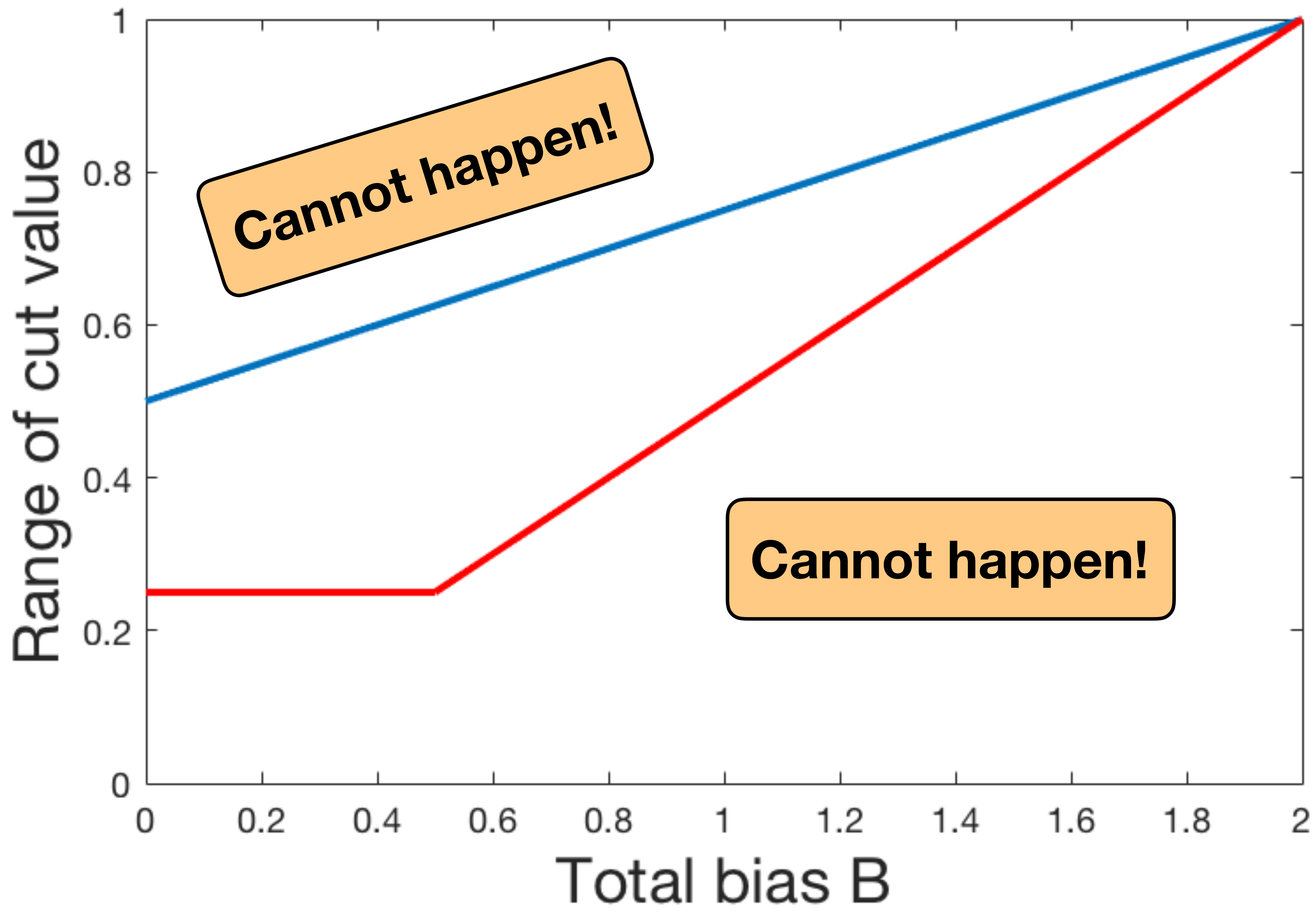

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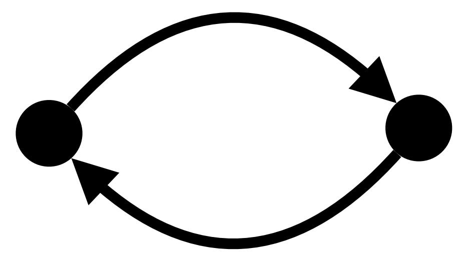
  Small bias => ∃ 

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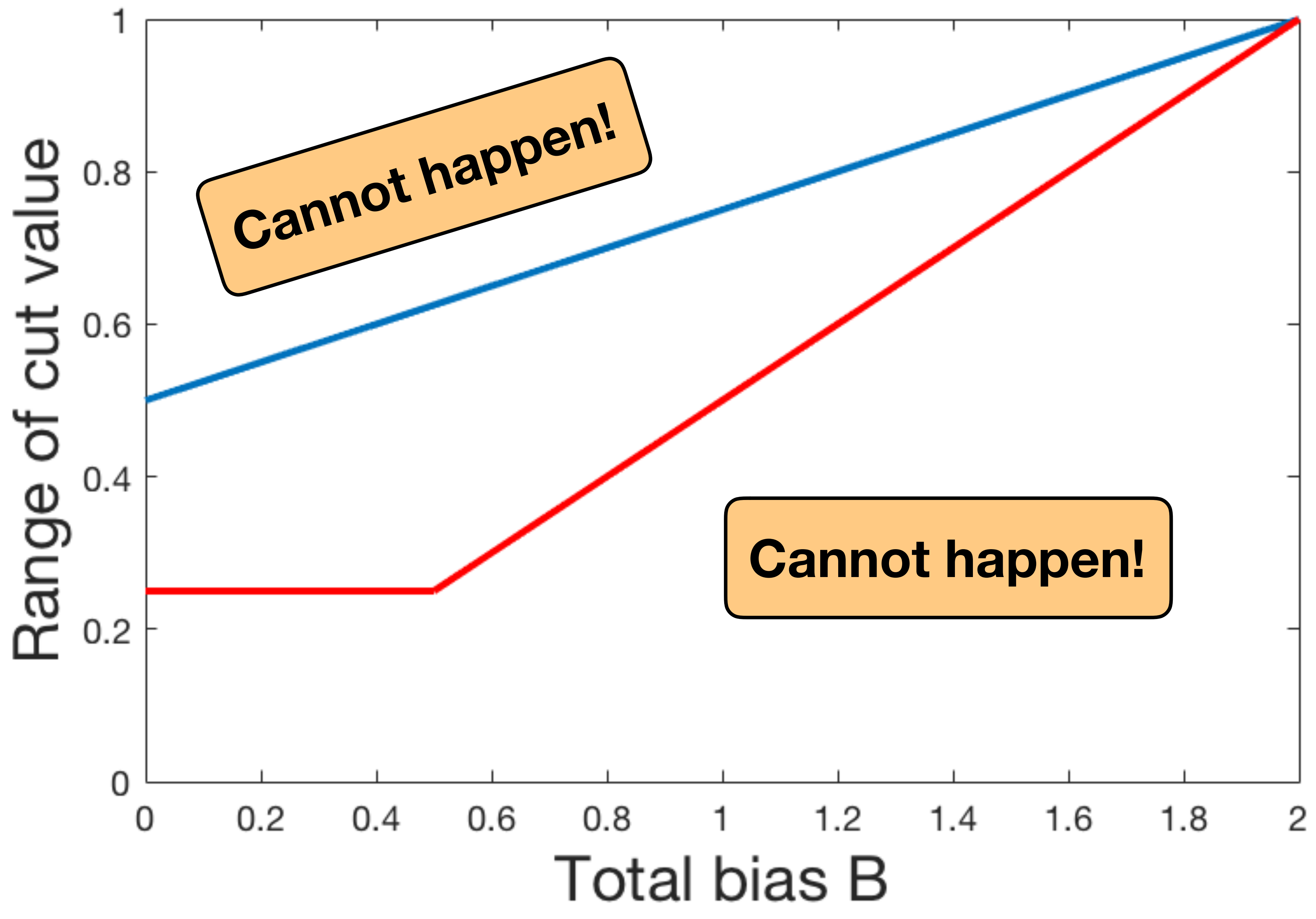

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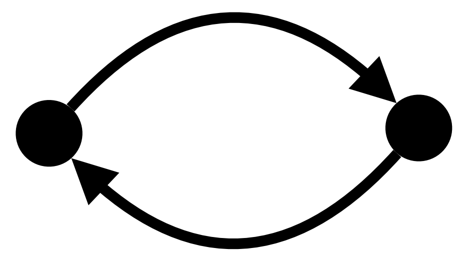
  Small bias => $\exists$ 

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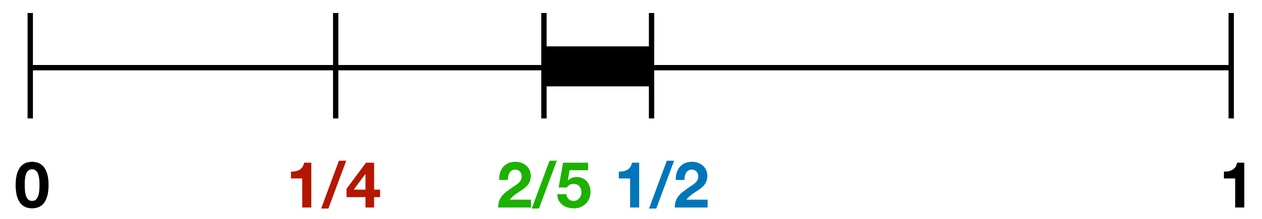

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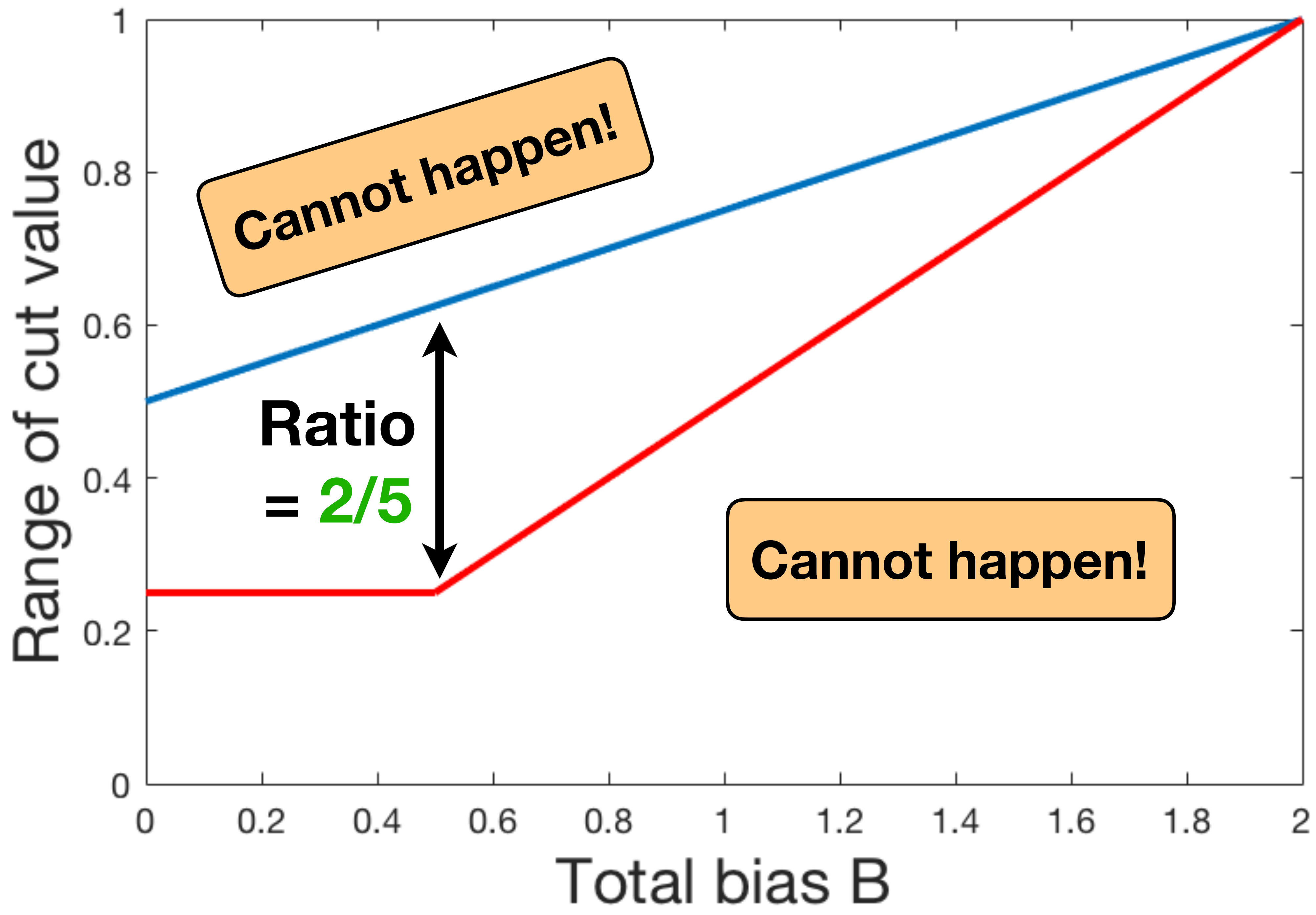

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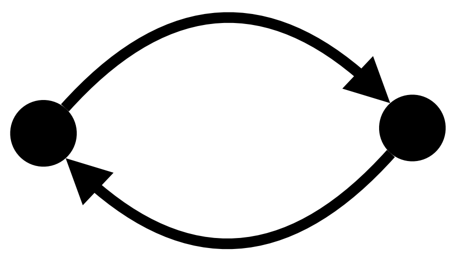
  Small bias => $\exists$ 

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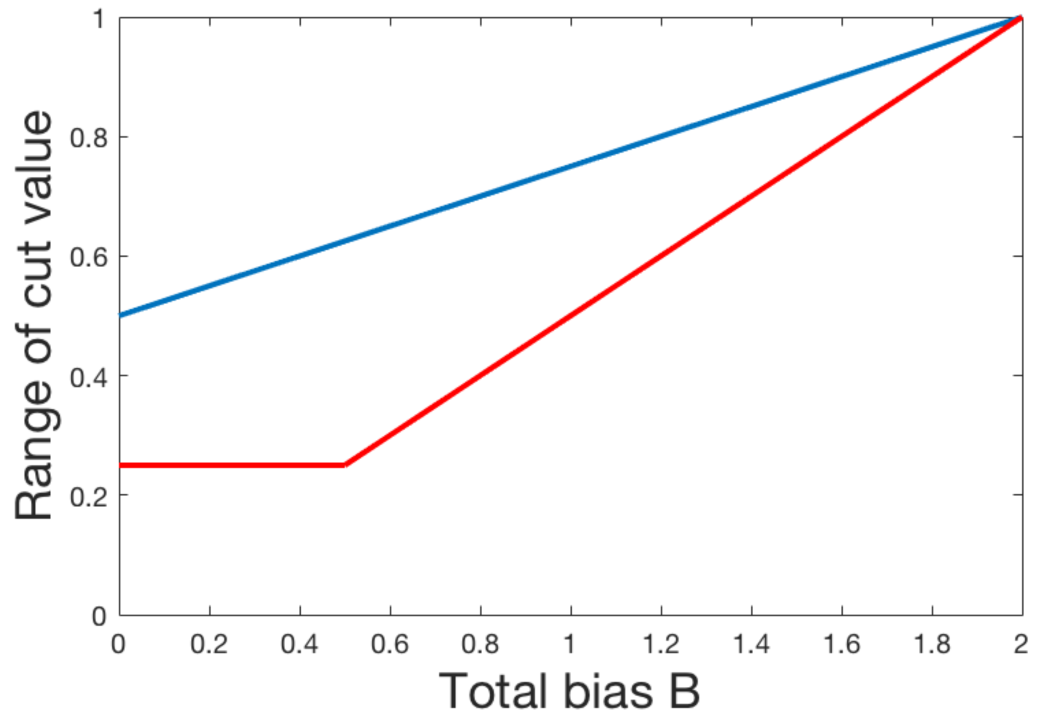

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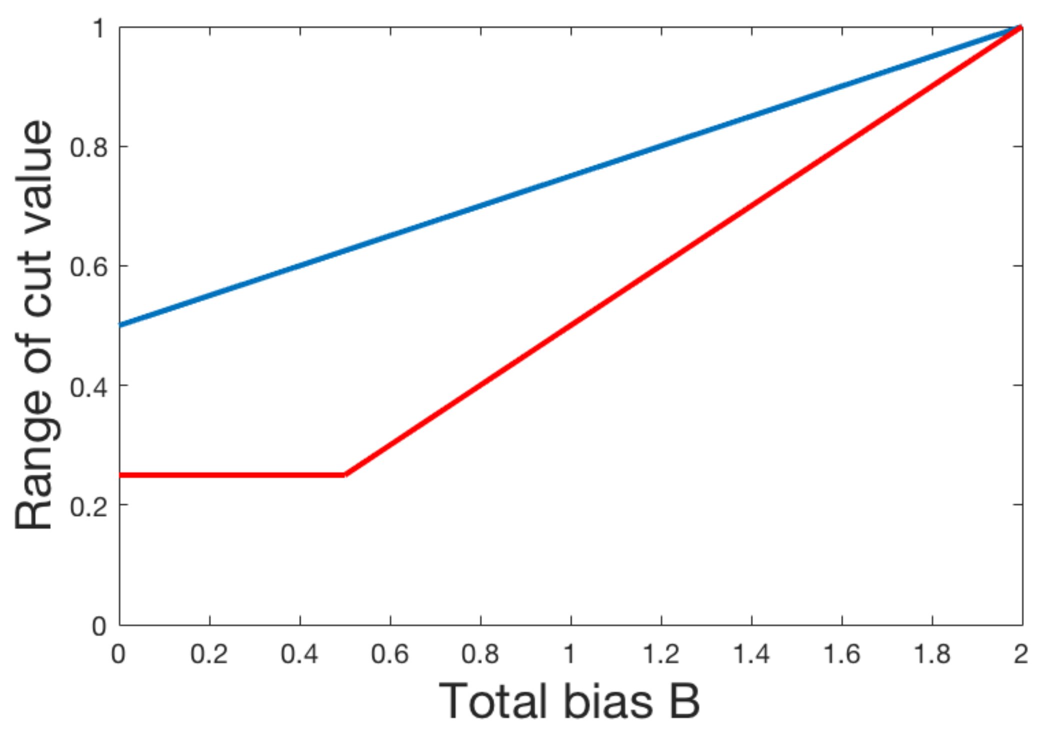

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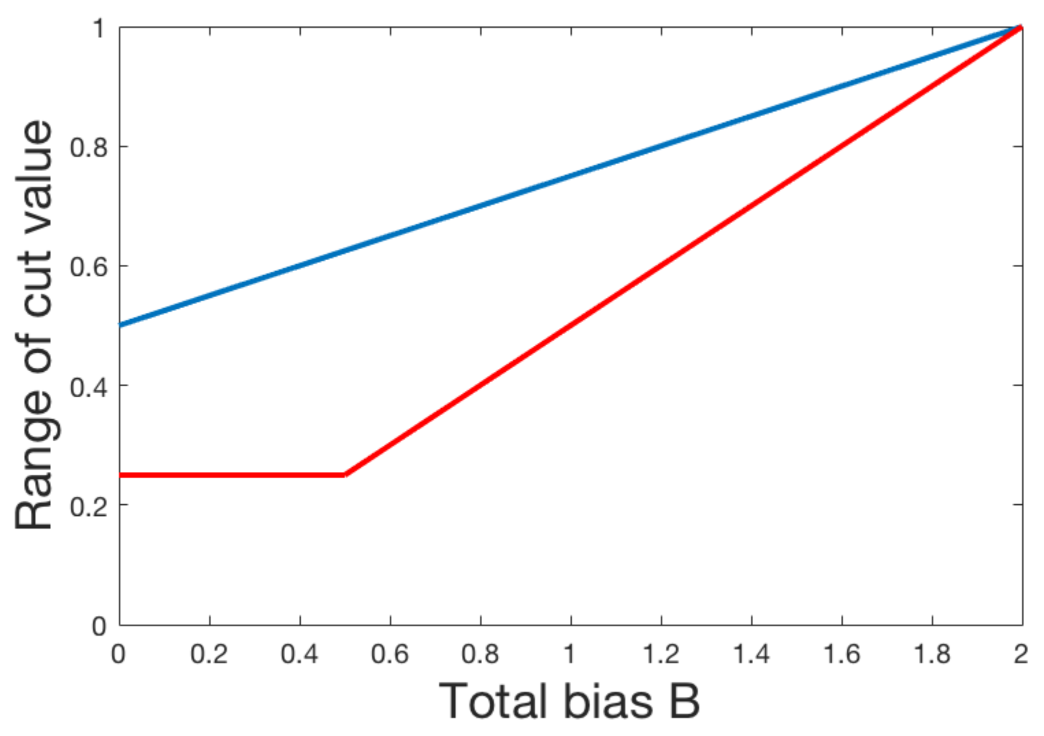

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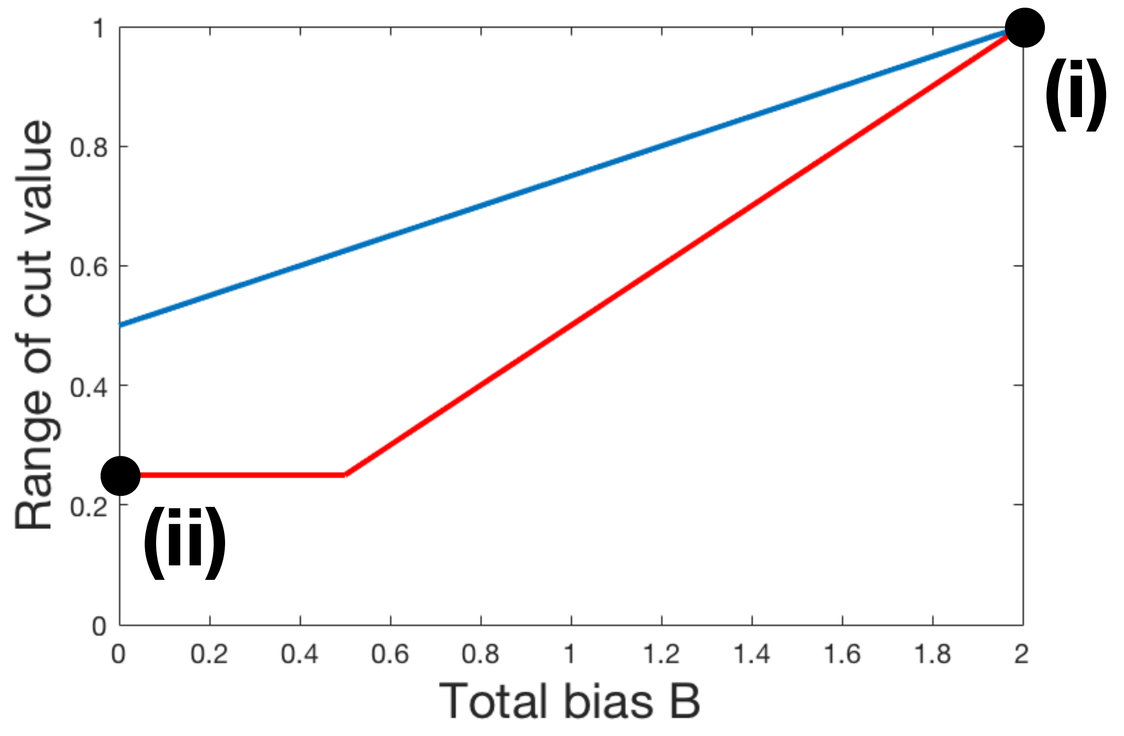

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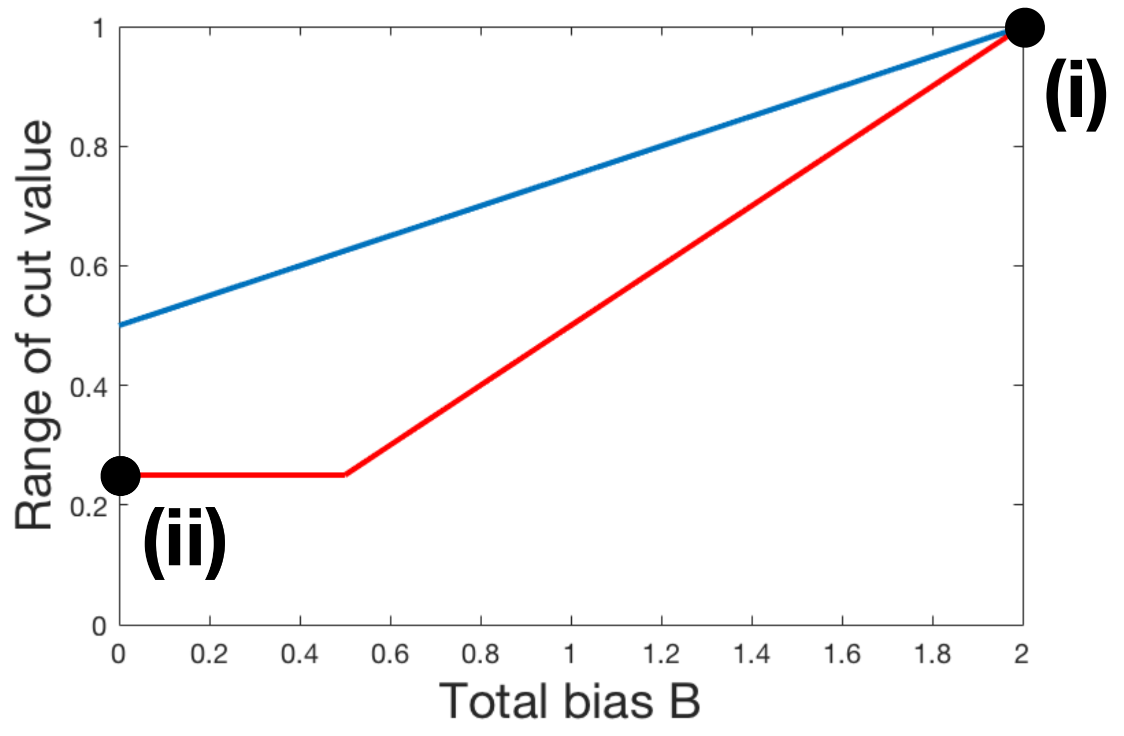

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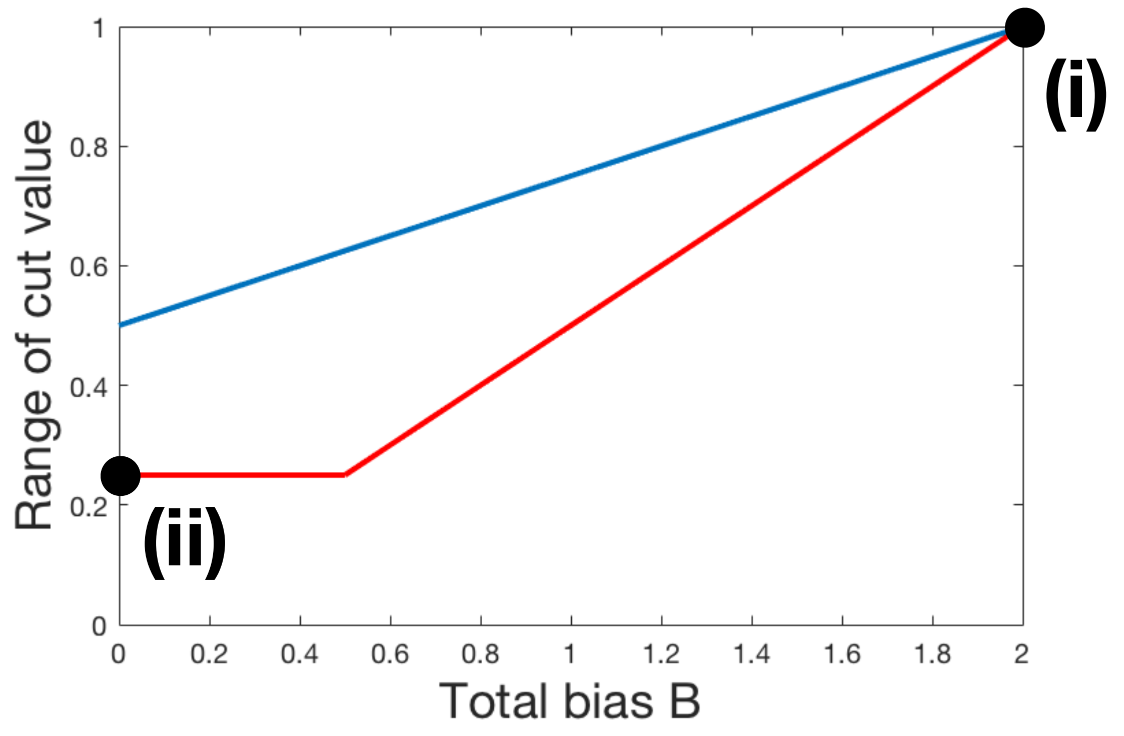

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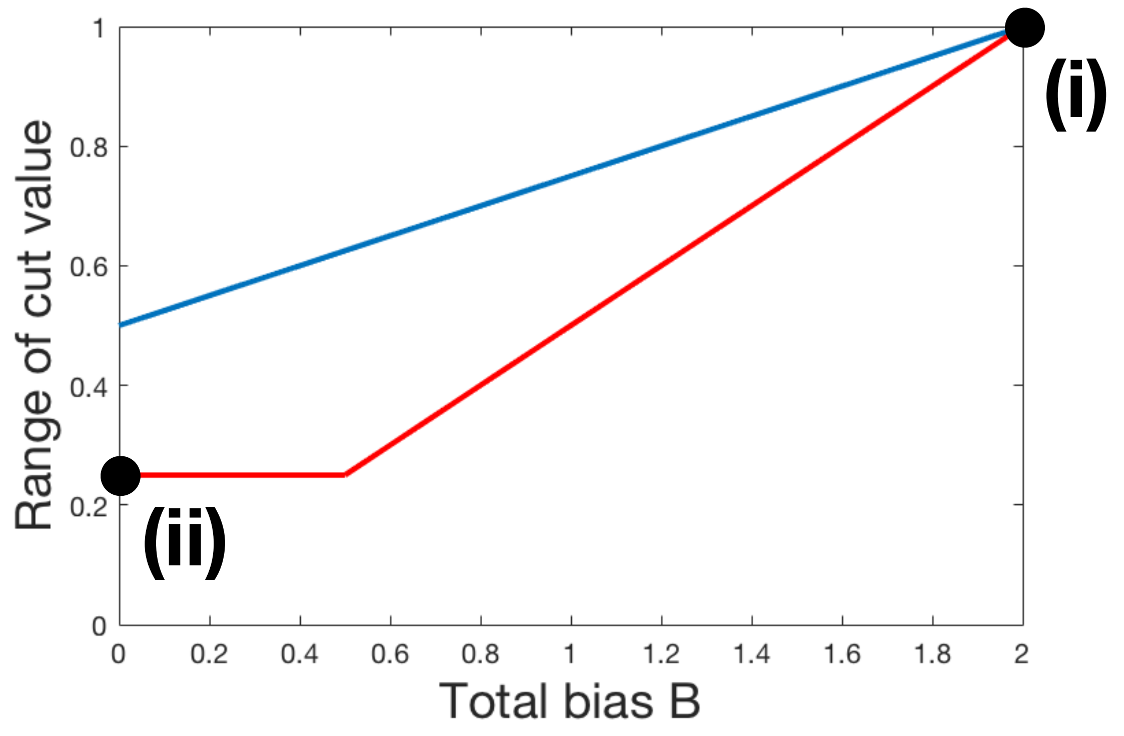

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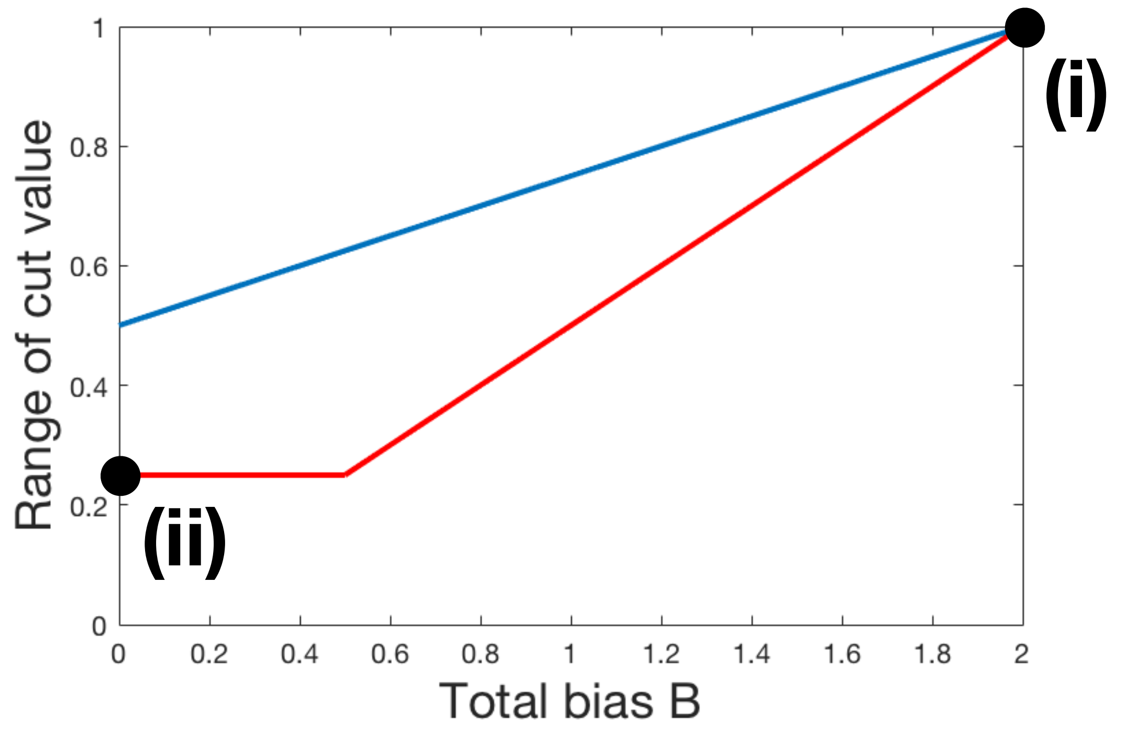

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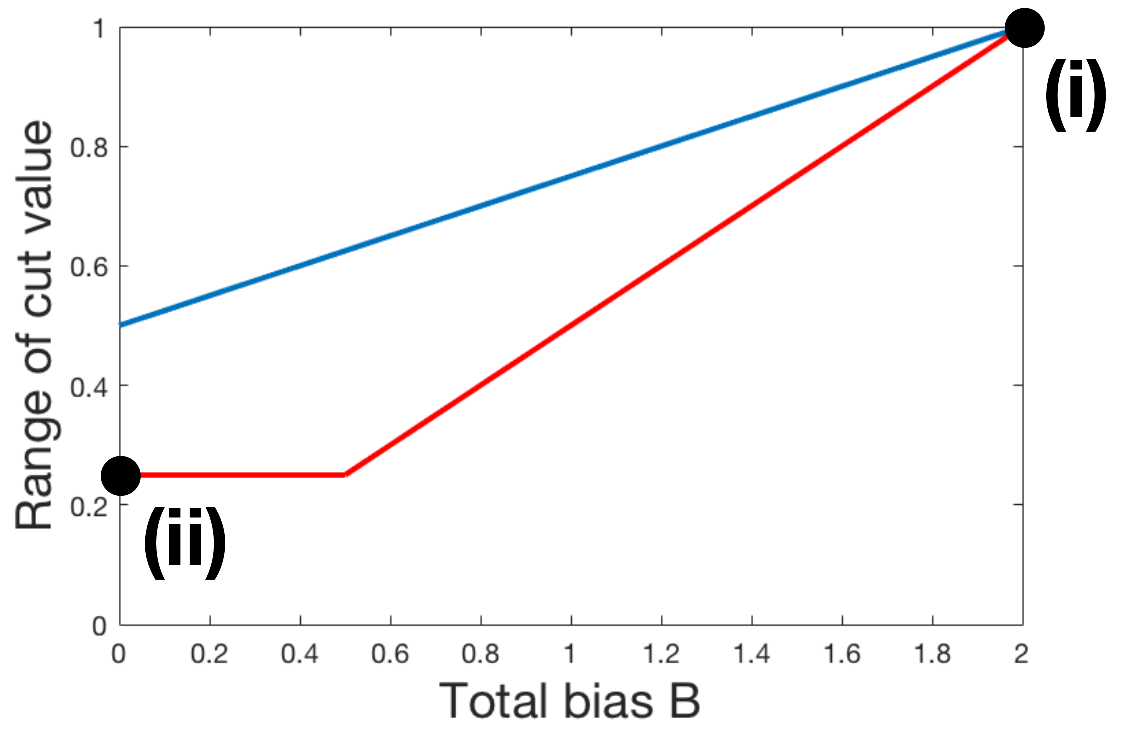

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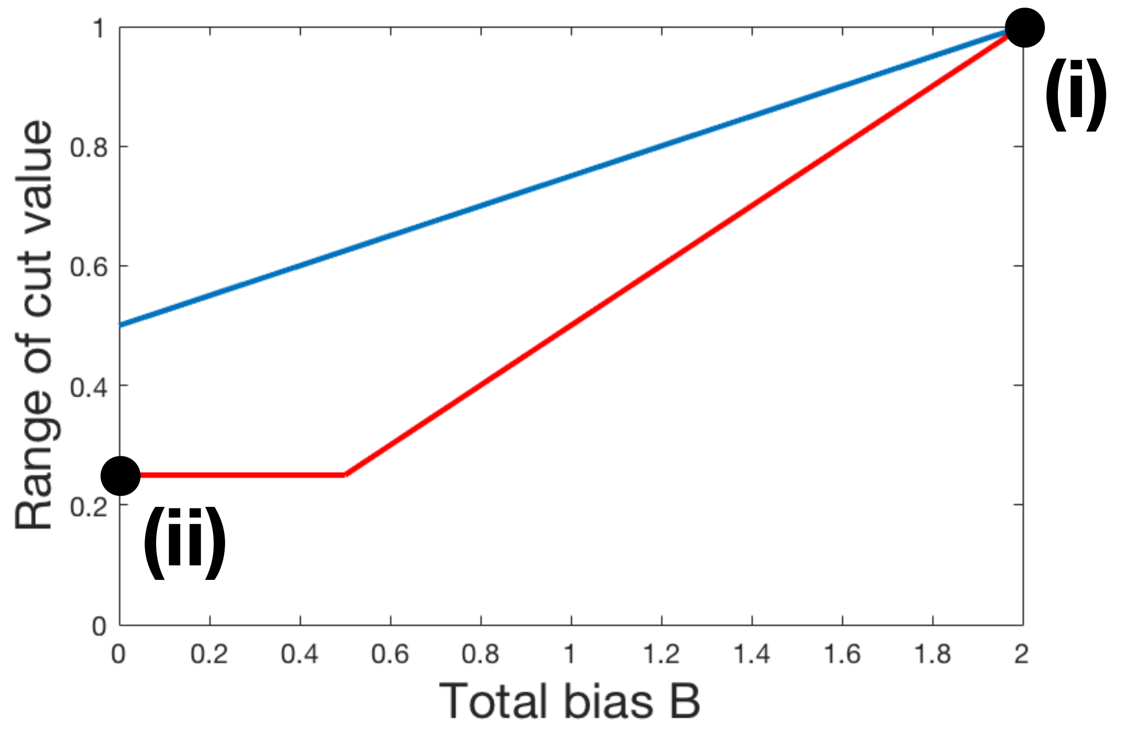

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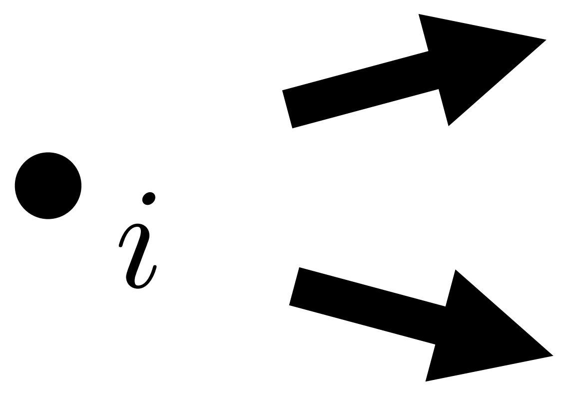

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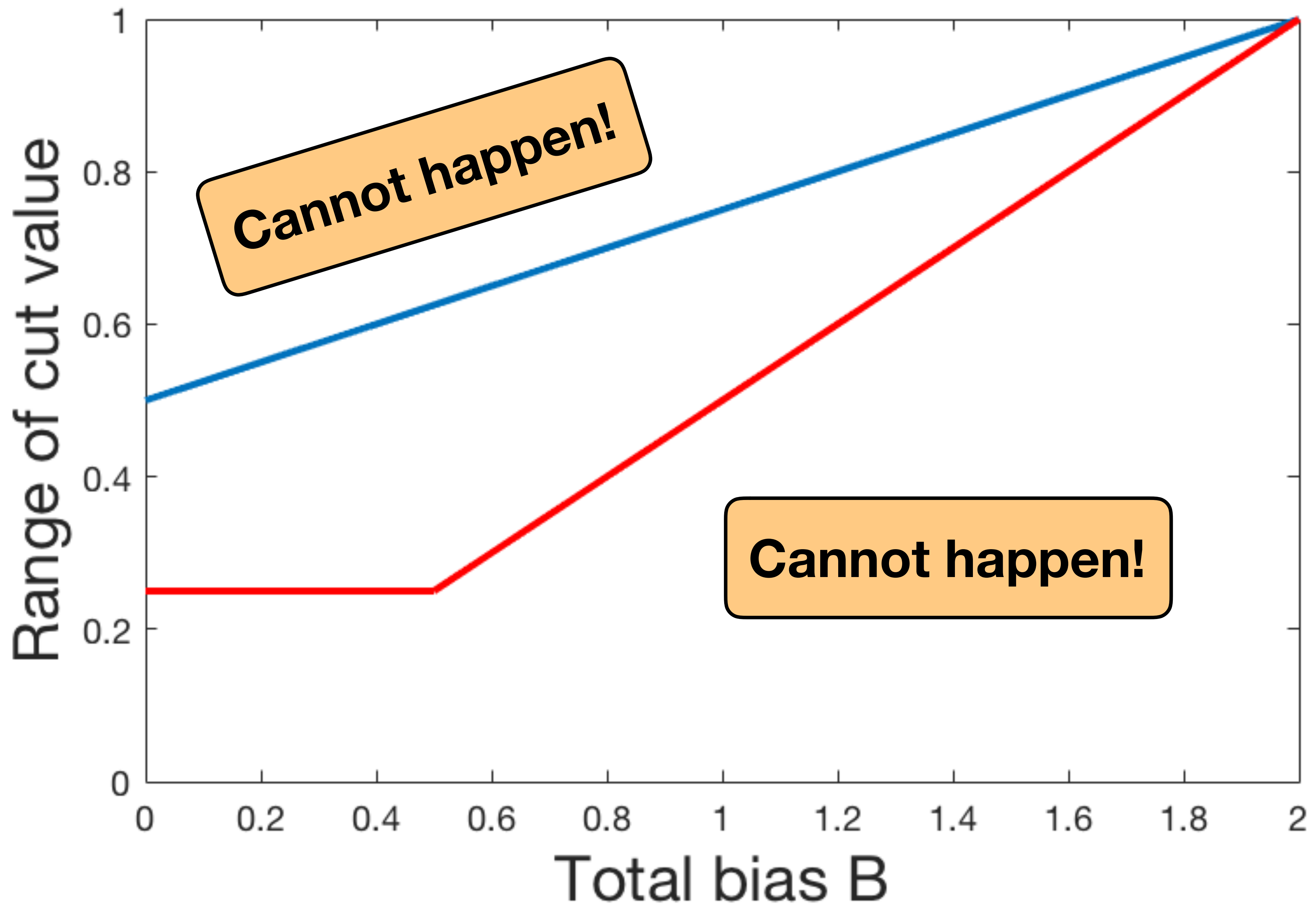

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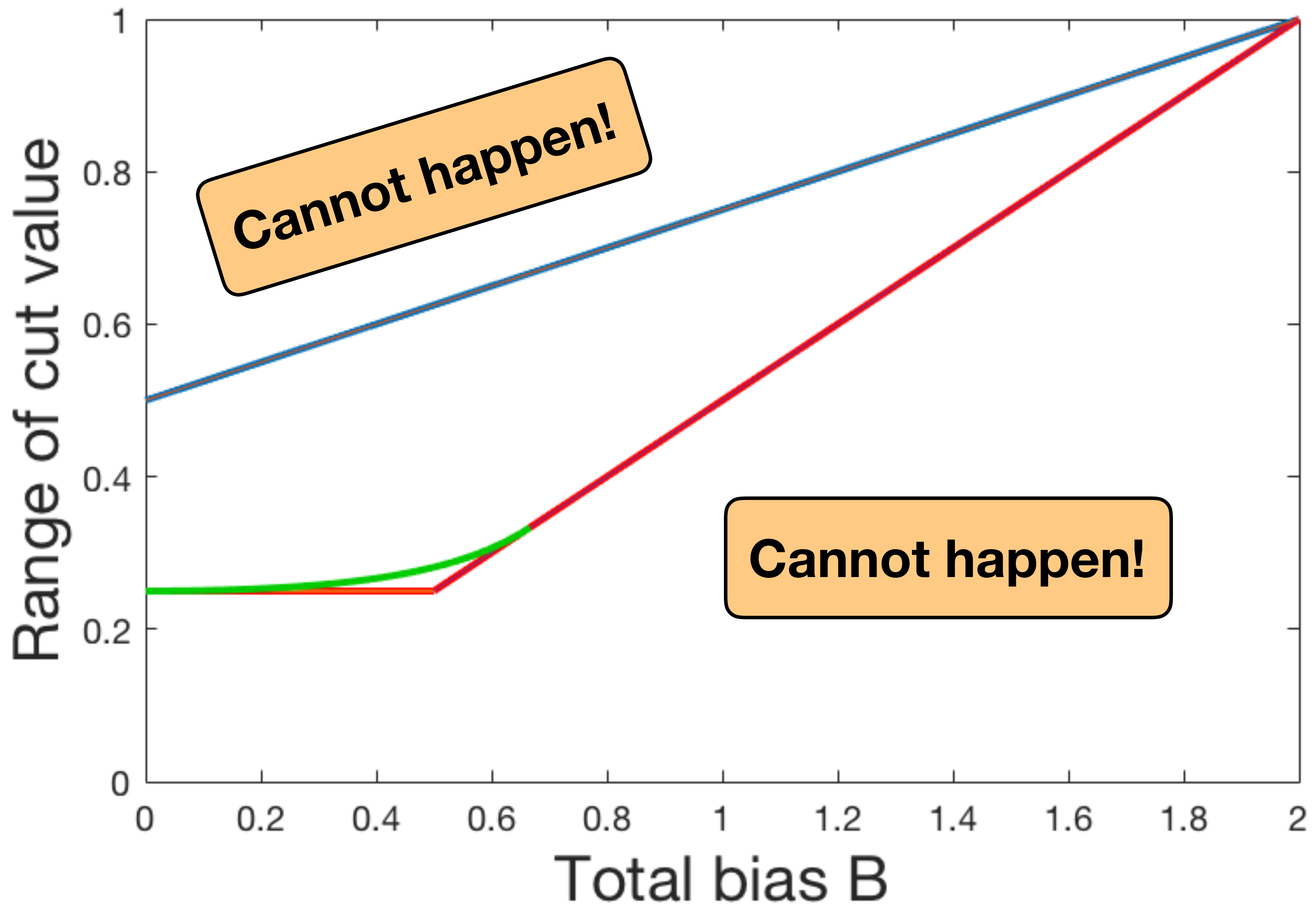

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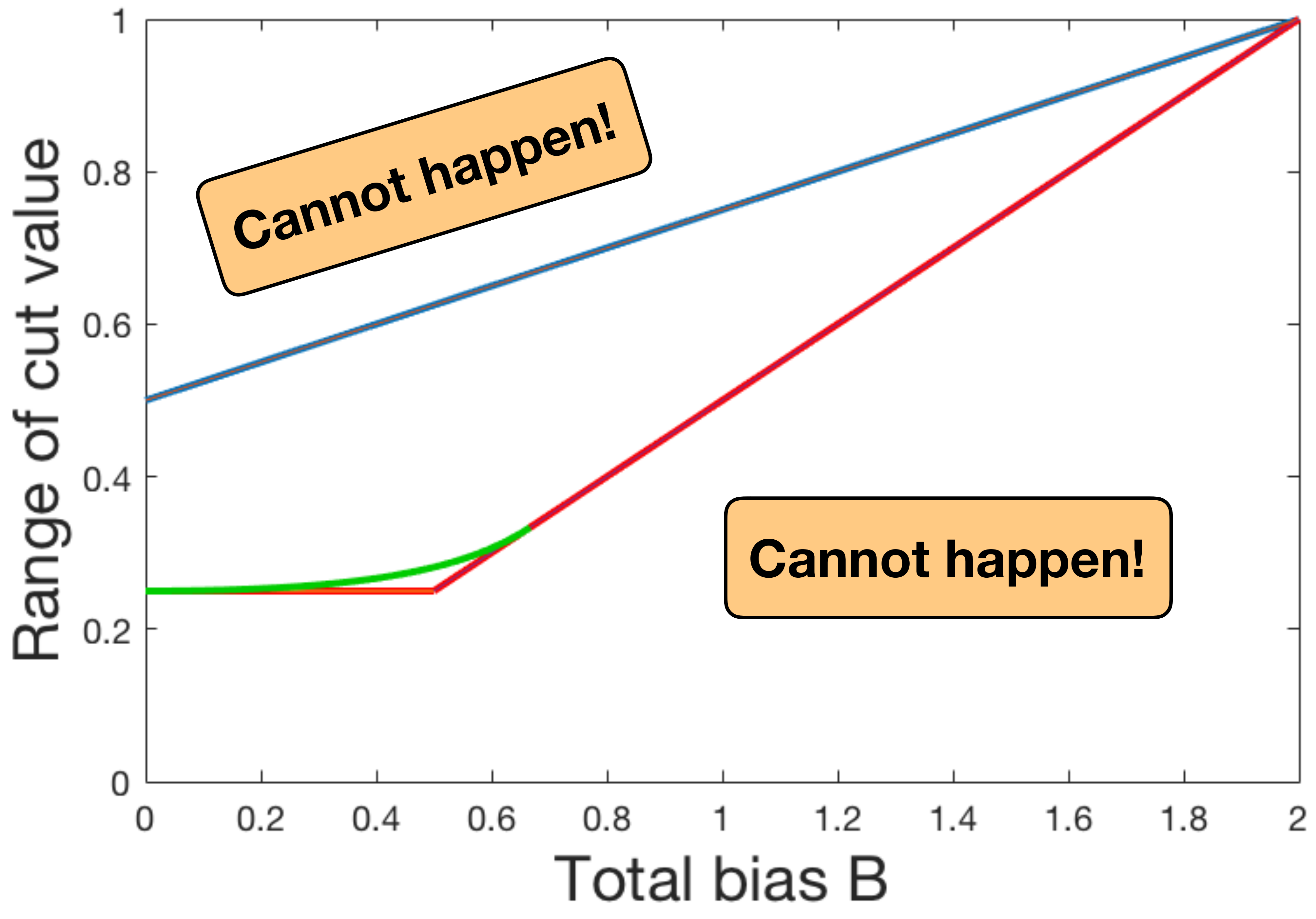

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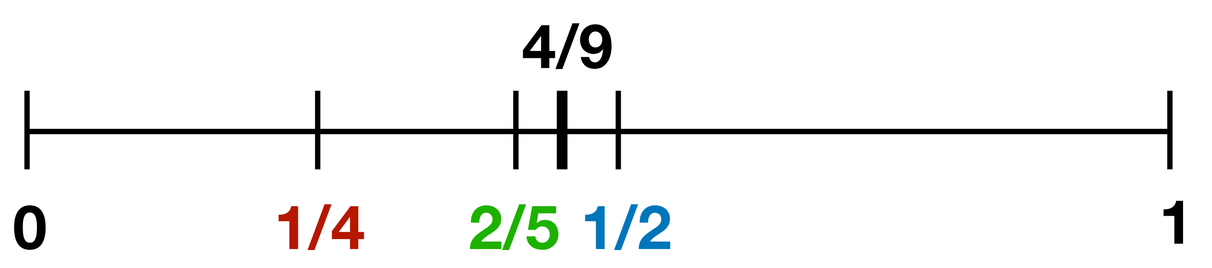

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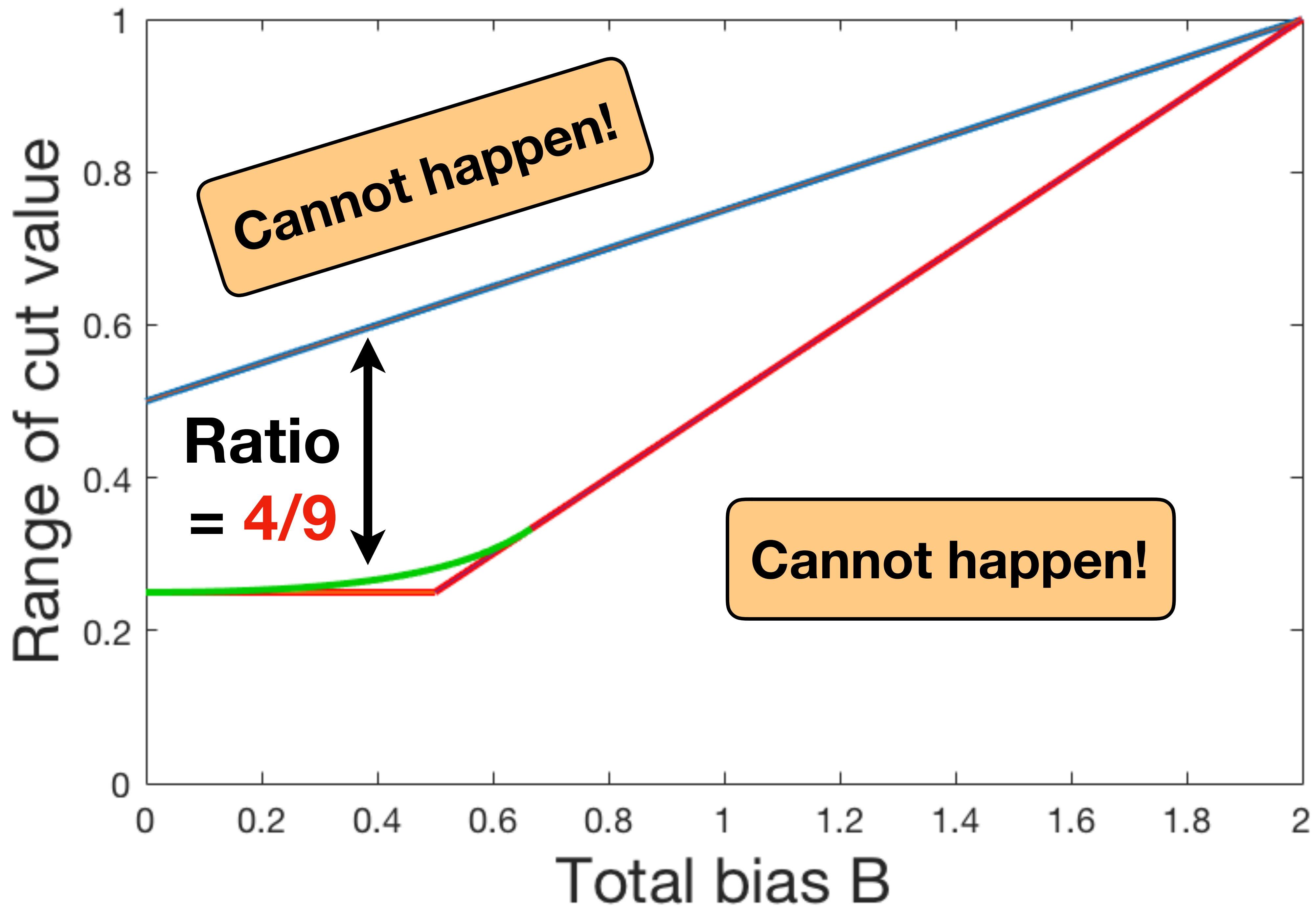

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- **Ratio**: When B = 2/5, the ratio is **4/9**.

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2EOR	$\frac{3}{4}$	$\left[\frac{3}{4}, 1\right]$	Trivial
2AND	$\frac{4}{9}$	$\left[\frac{2}{5}, \frac{1}{2}\right]$	Biased sampling

Local Random Sampling is Optimal in Streaming Model

- It turns out that the optimal approx. ratio of all the boolean 2CSP can be achieved by local random sampling analysis.

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- See our paper for more details!

Hardness

Hardness

Find Instances Matching
Random Sampling's Bounds

Streaming Lower Bounds via Communication Complexity

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- *Unconditional lower bounds* from **communication games**.

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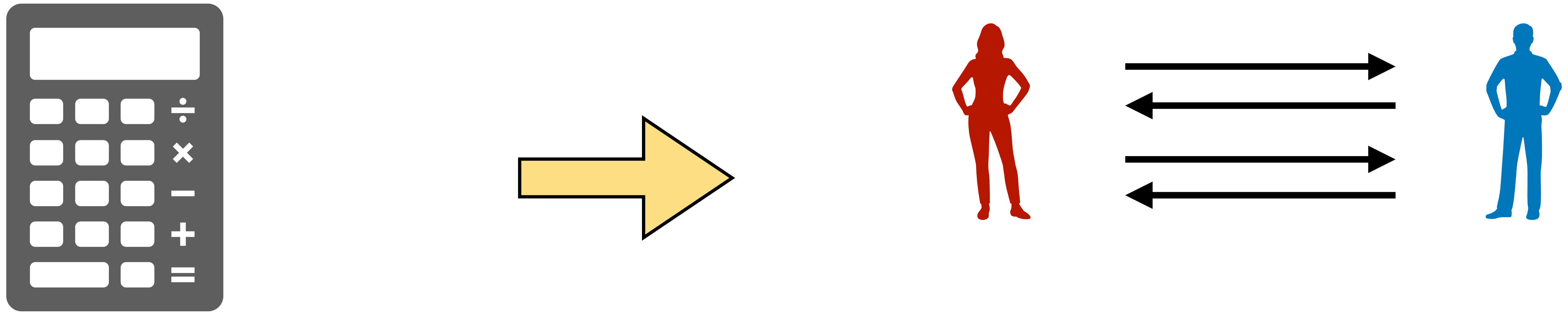

Streaming Algorithm

Communication Protocol

Streaming Lower Bounds via Communication Complexity

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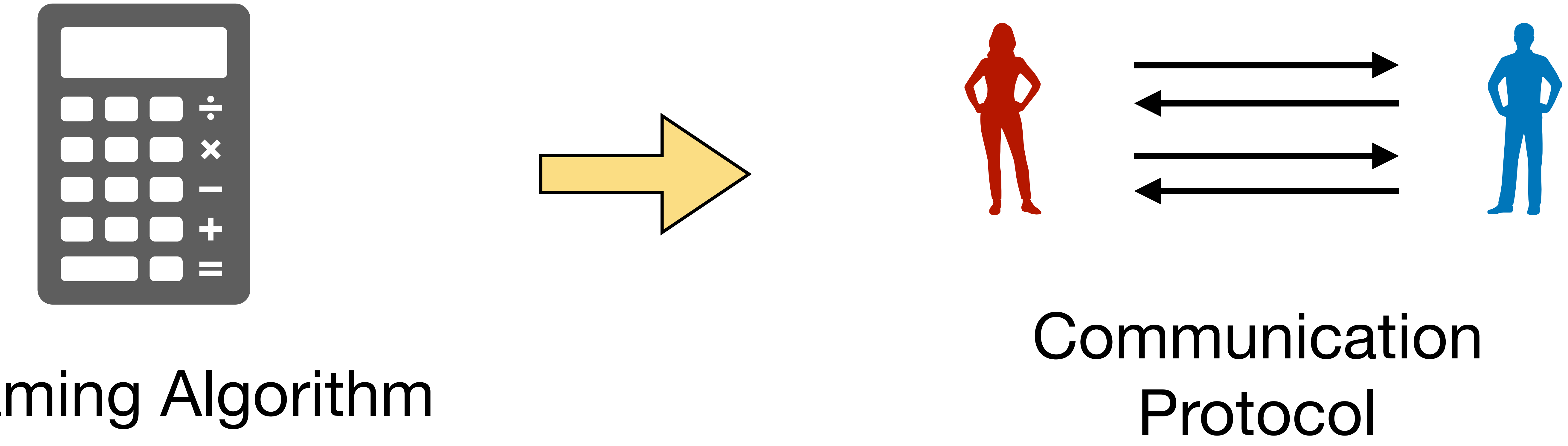

- **Usage**: Alice and Bob insert some inputs to the streaming algorithm and send the "***configuration***" as the message.

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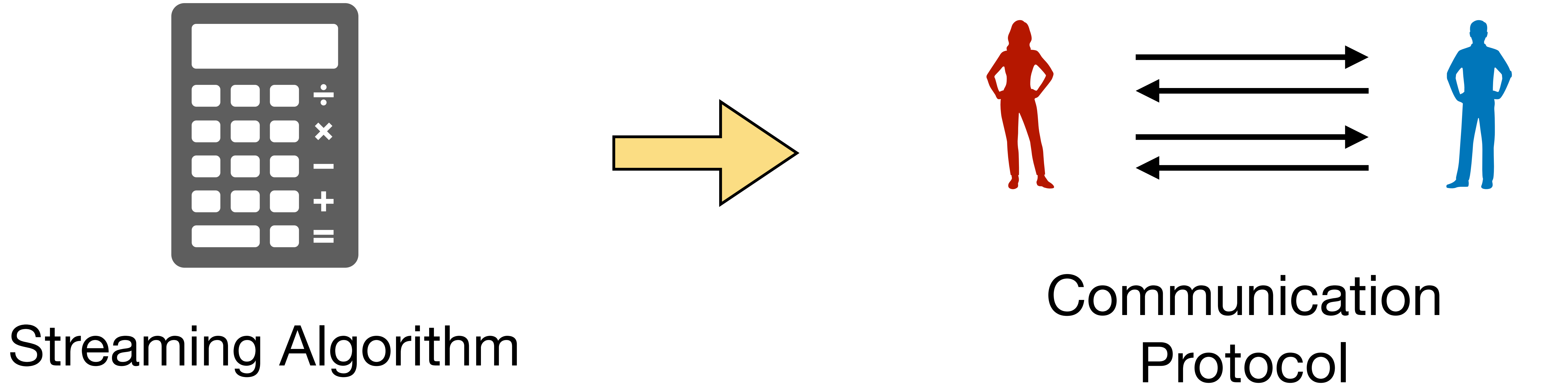

- **Usage**: Alice and Bob insert some inputs to the streaming algorithm and send the “***configuration***” as the message.
- Space complexity of streaming algorithm **>=** communication complexity.

Distributional Boolean Hidden Partition (DBHP) Problem

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Alice

$$X^* \in \{0,1\}^n$$

Distributional Boolean Hidden Partition (DBHP) Problem

* Each row of M contains exactly two 1s.

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$X^* \in \{0,1\}^n$

c bits →

Bob

$w \in \{0,1\}^{0.01n}$

$M \in \{0,1\}^{0.01n \times n}$

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→ Yes

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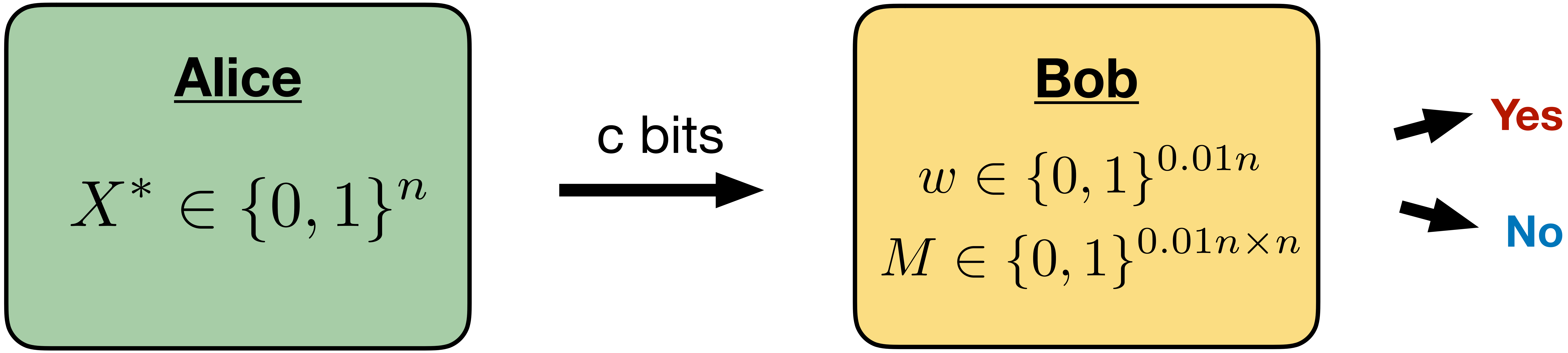

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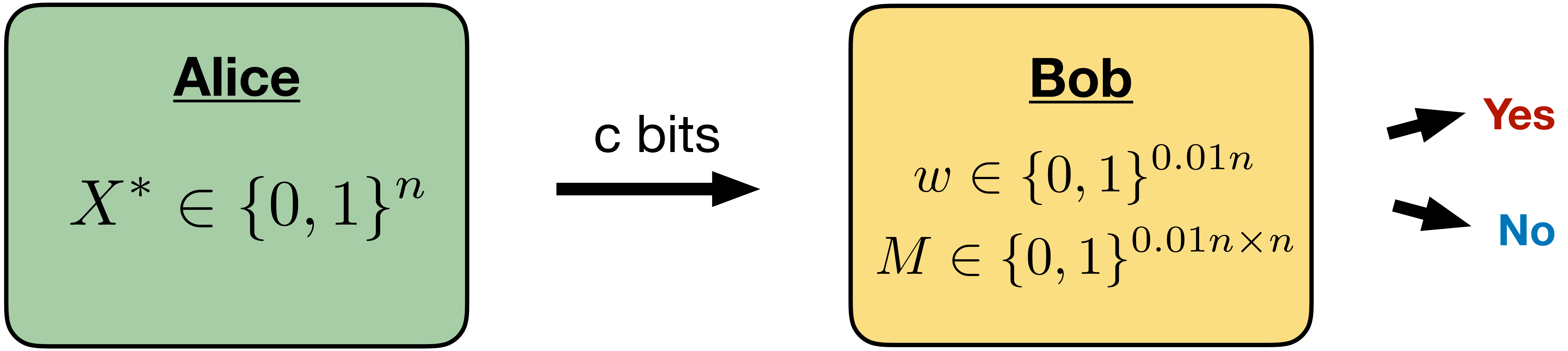

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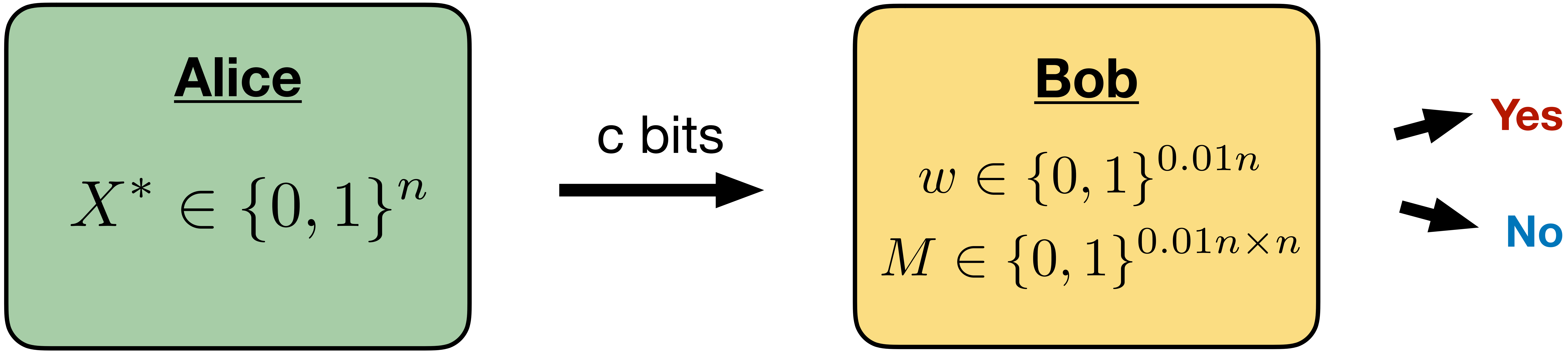

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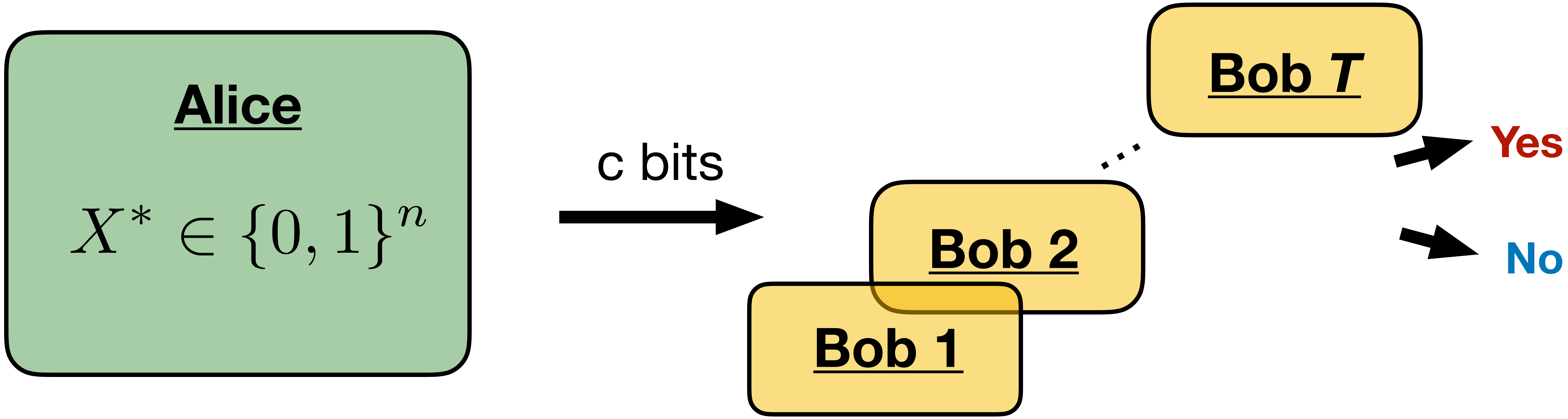

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- **Parallel repetition:** constant many copies to increase the number of edges.

Example of DBHP (with Parallel Repetition)

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Bob 1

$$w_1 = [1\ 0\ 0]^\top$$

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

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- Can you see the connection to Max-CUT?

Reducing DBHP to Max-CUT

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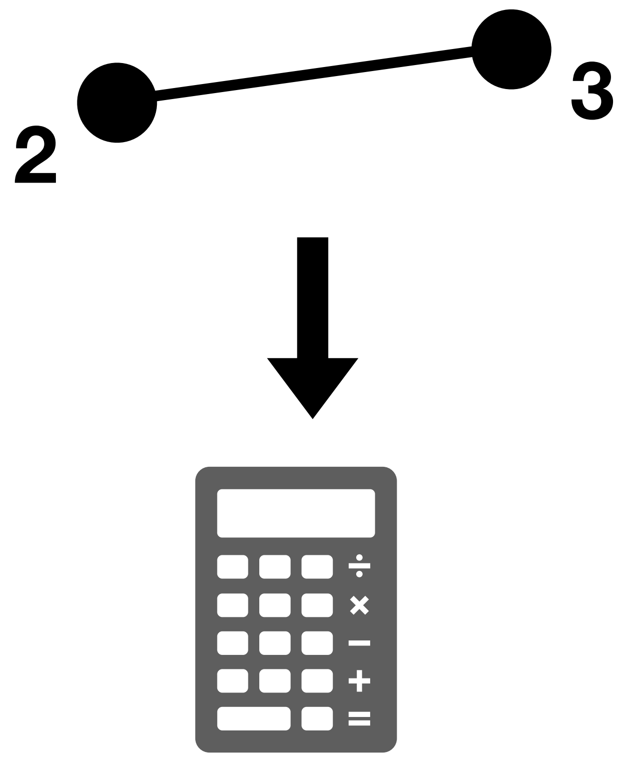

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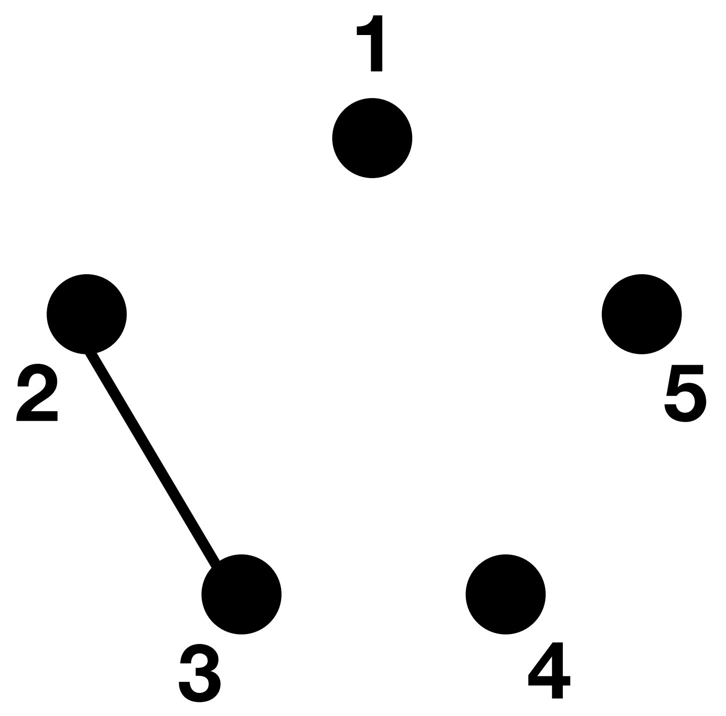

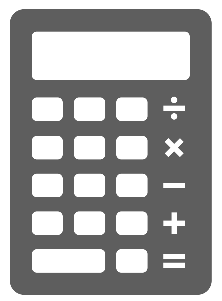

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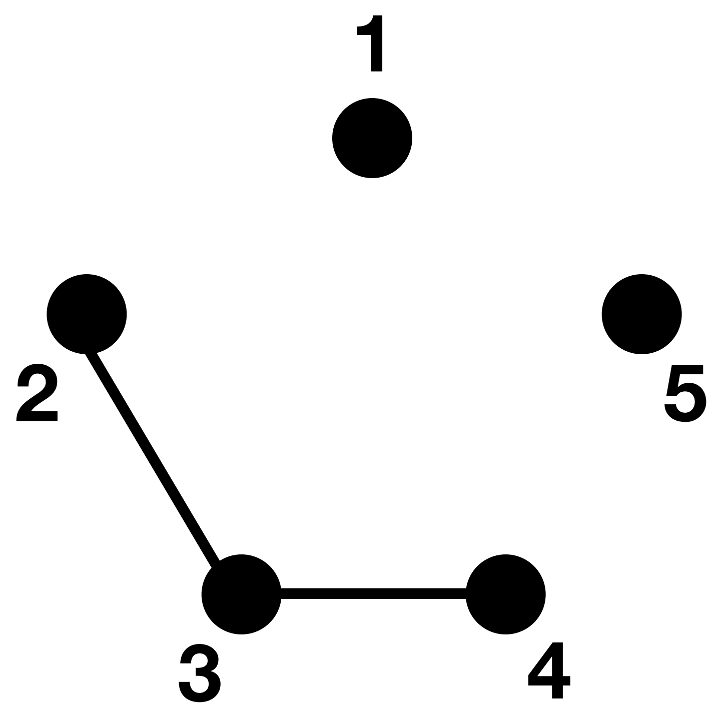

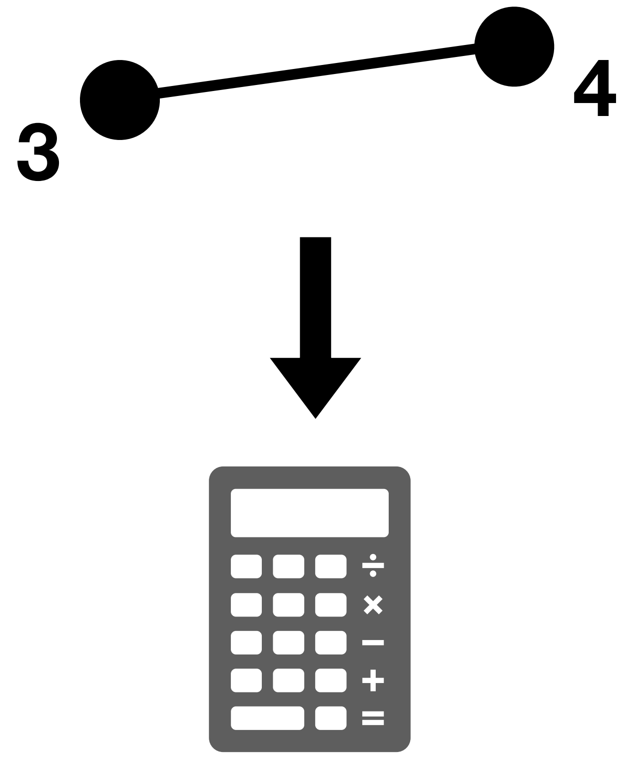

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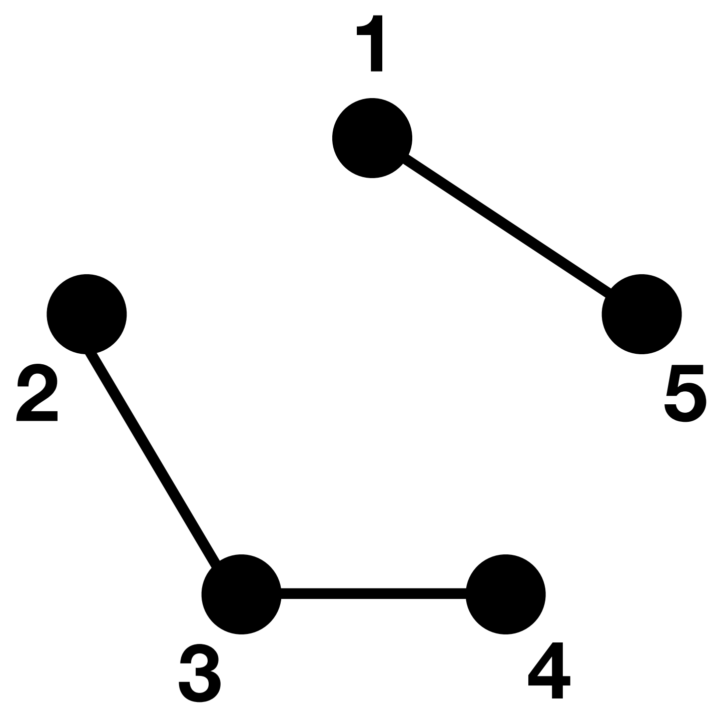

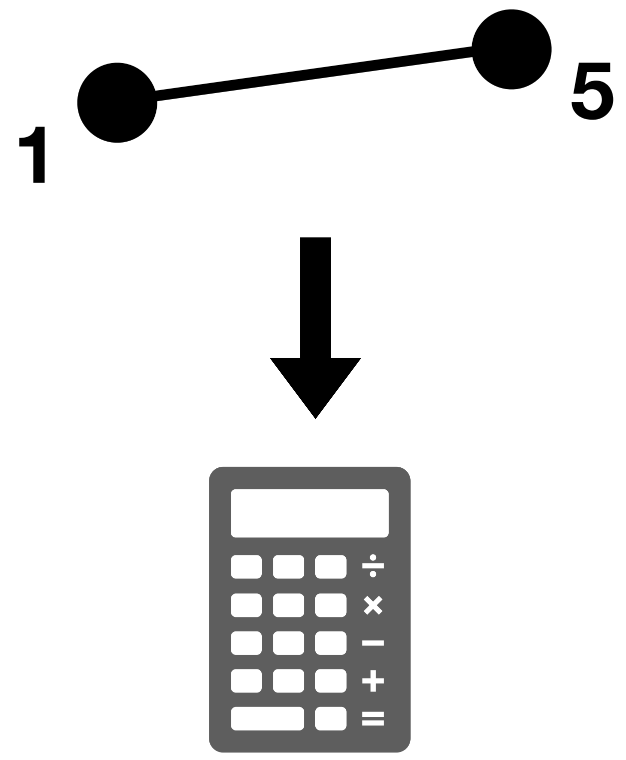

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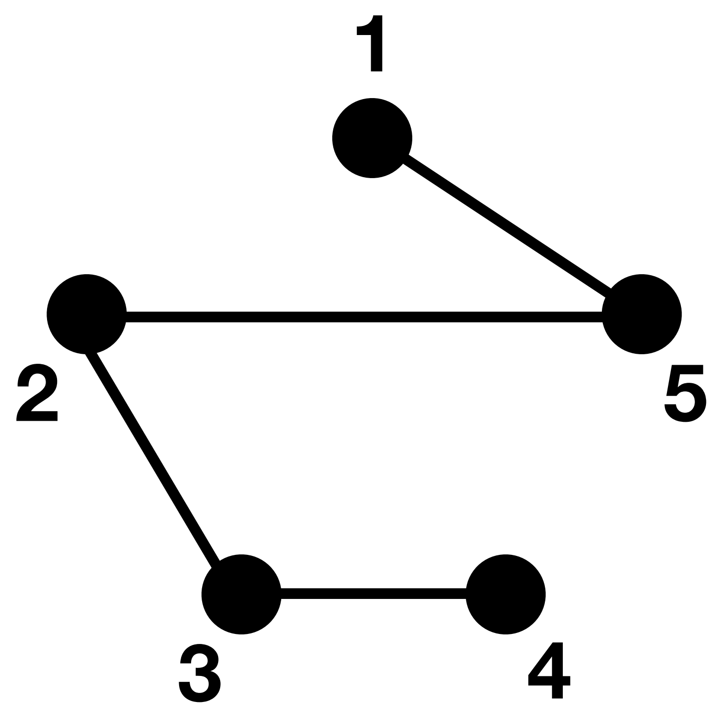

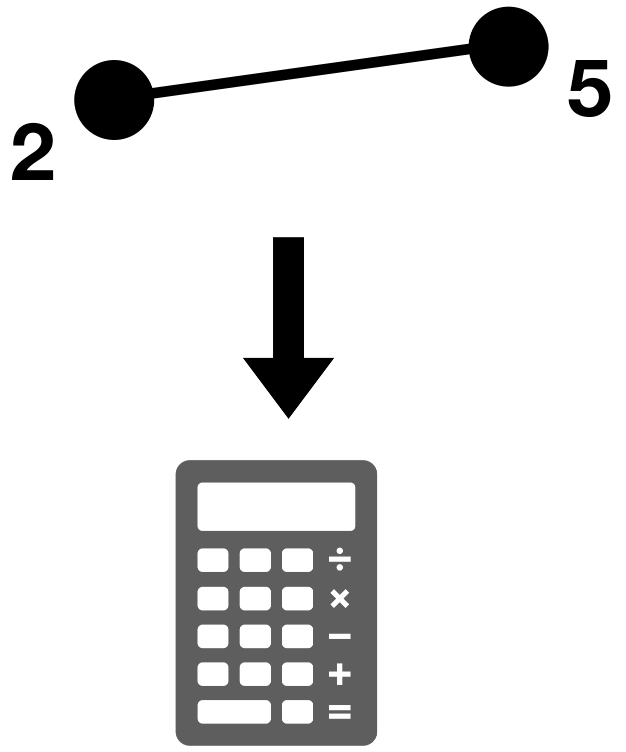

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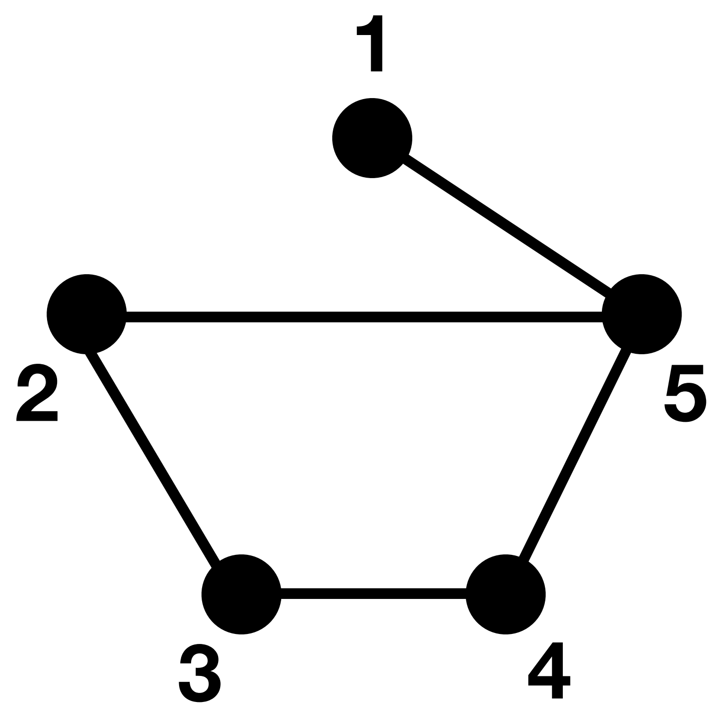

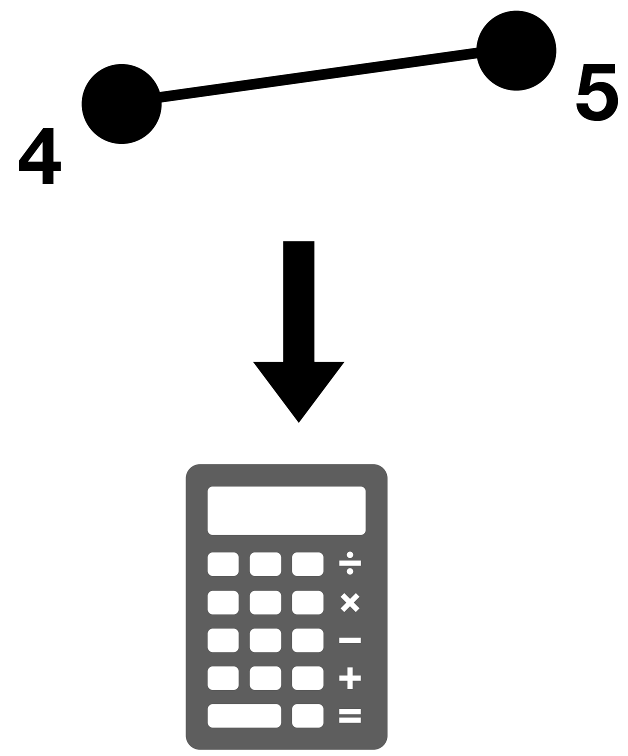

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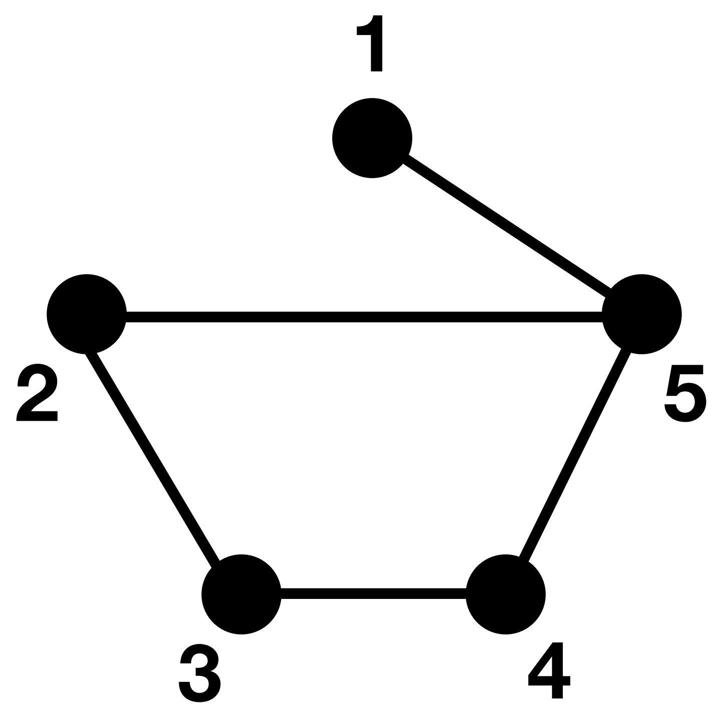

$\rightarrow v$

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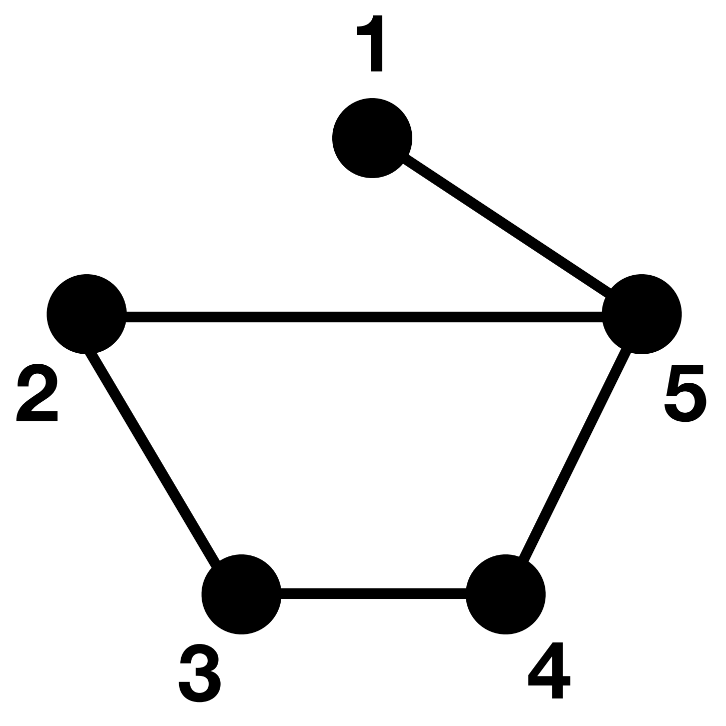

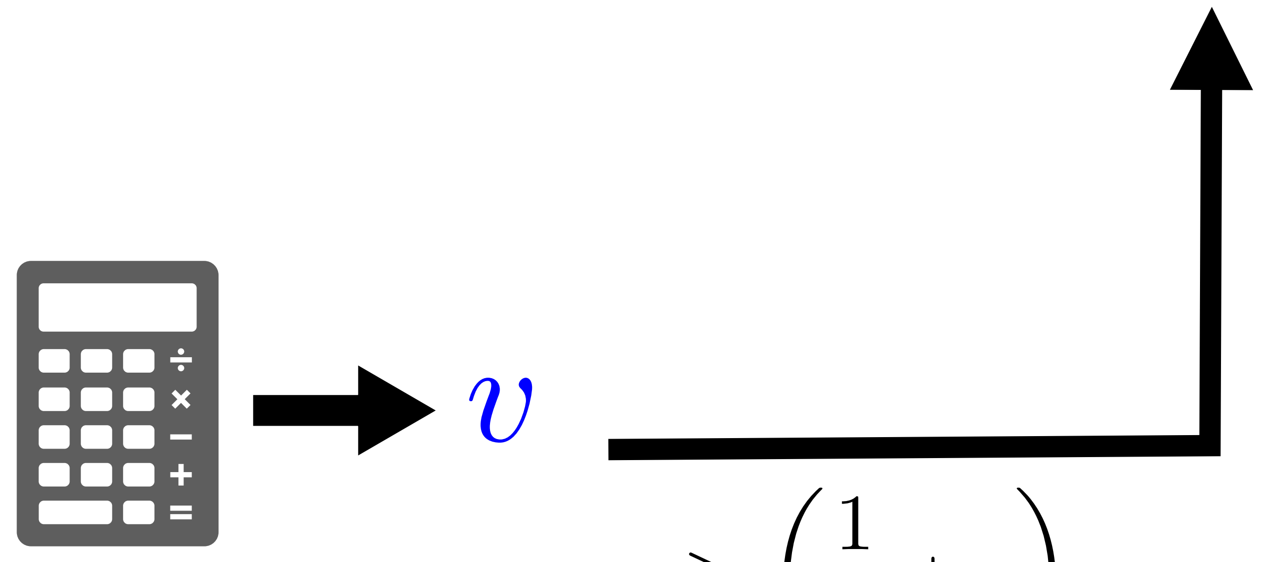

v

$v \geq \left(\frac{1}{2} + \epsilon\right) \cdot m$

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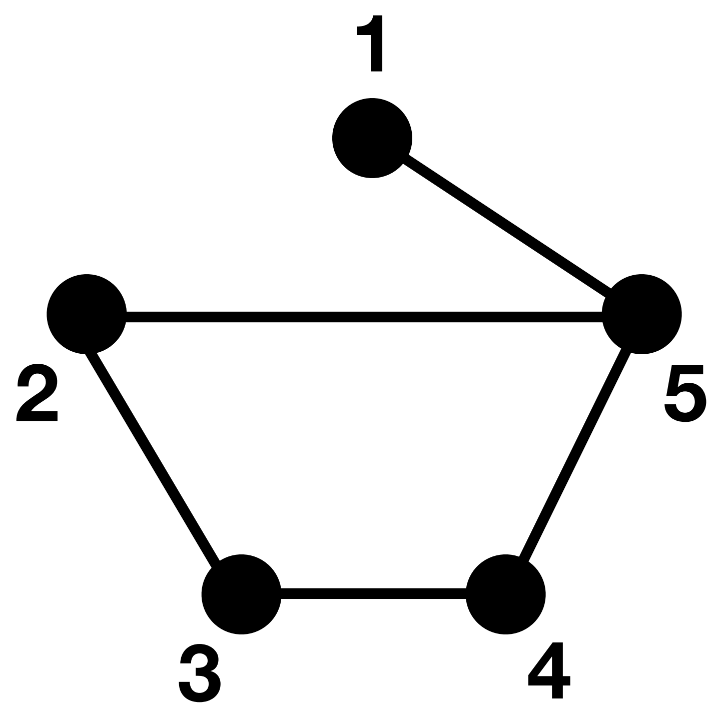

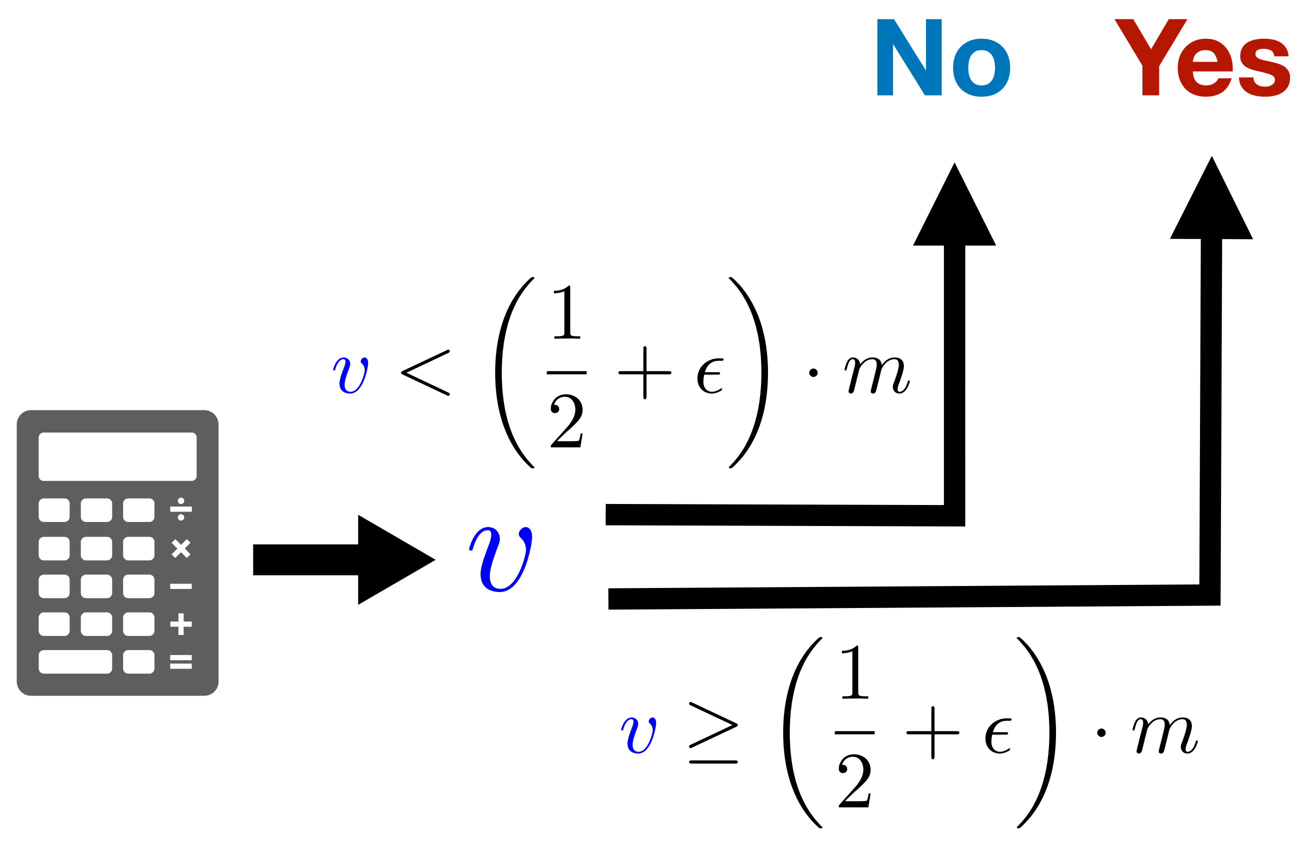

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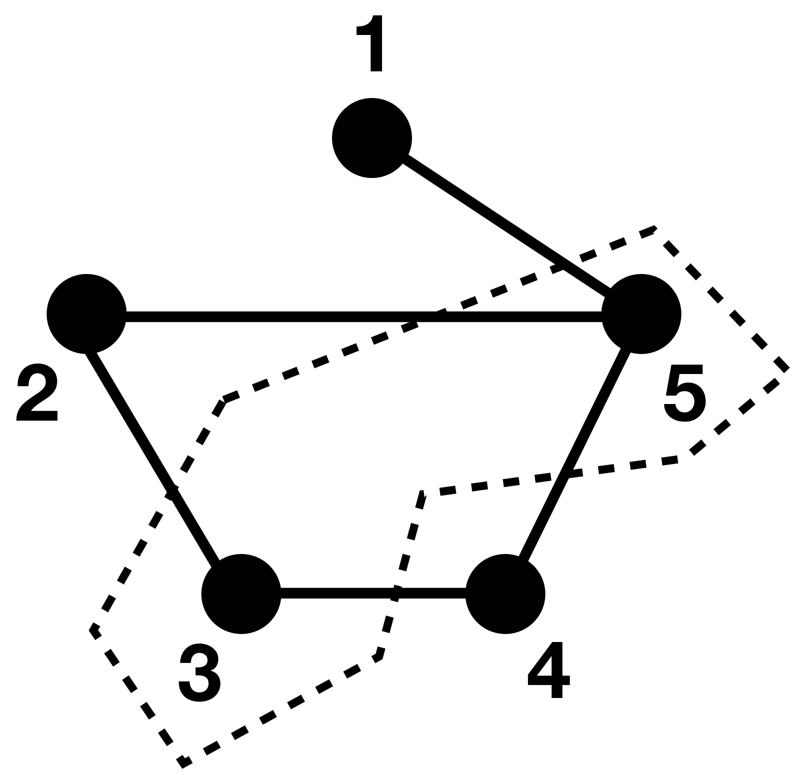

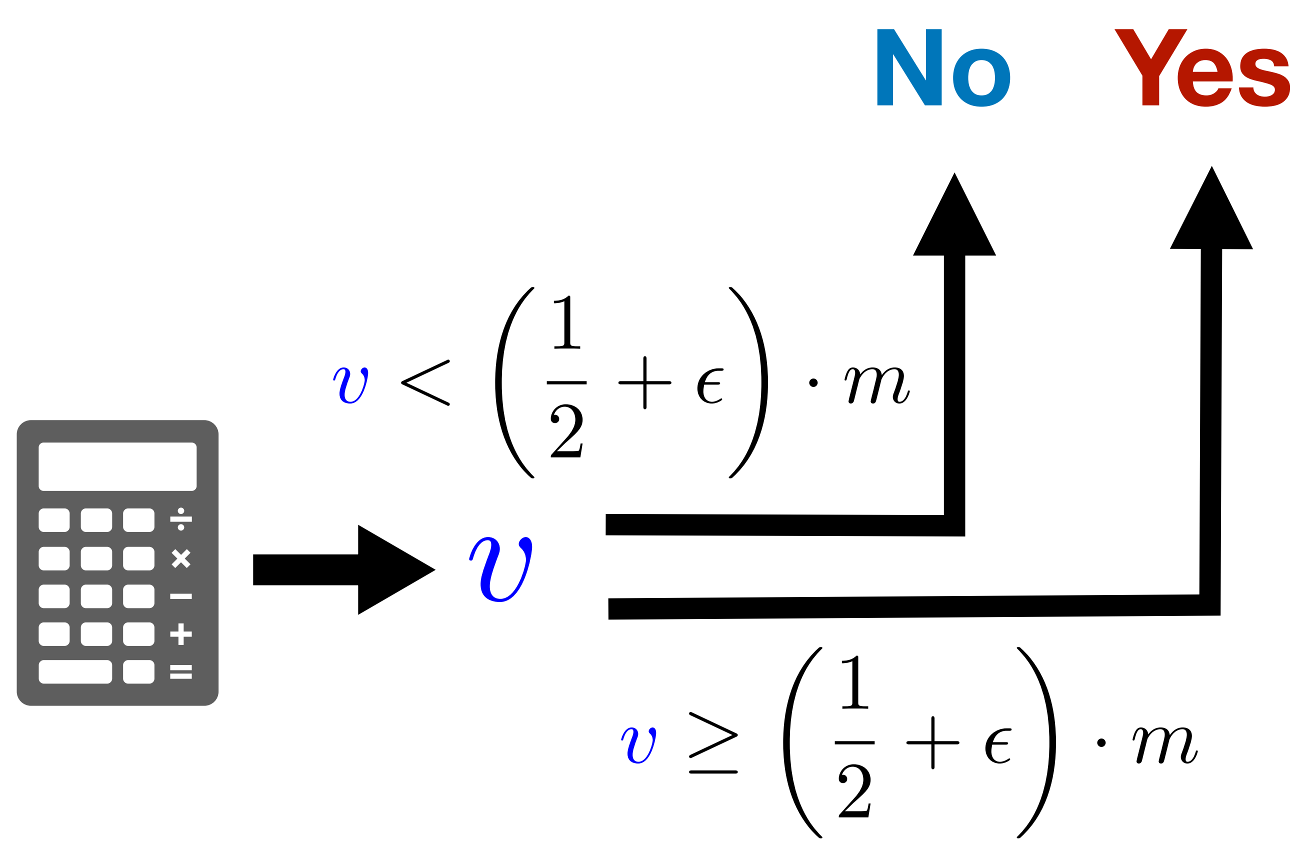

How to Use DBHP? A Graph View

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- Think of each row of M_t as a random edge and w_t picks the edges.

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Yes Distribution

No Distribution

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$\exists X^* \text{ s.t. } w_t = M_t X^*$

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w_t is uniformly random

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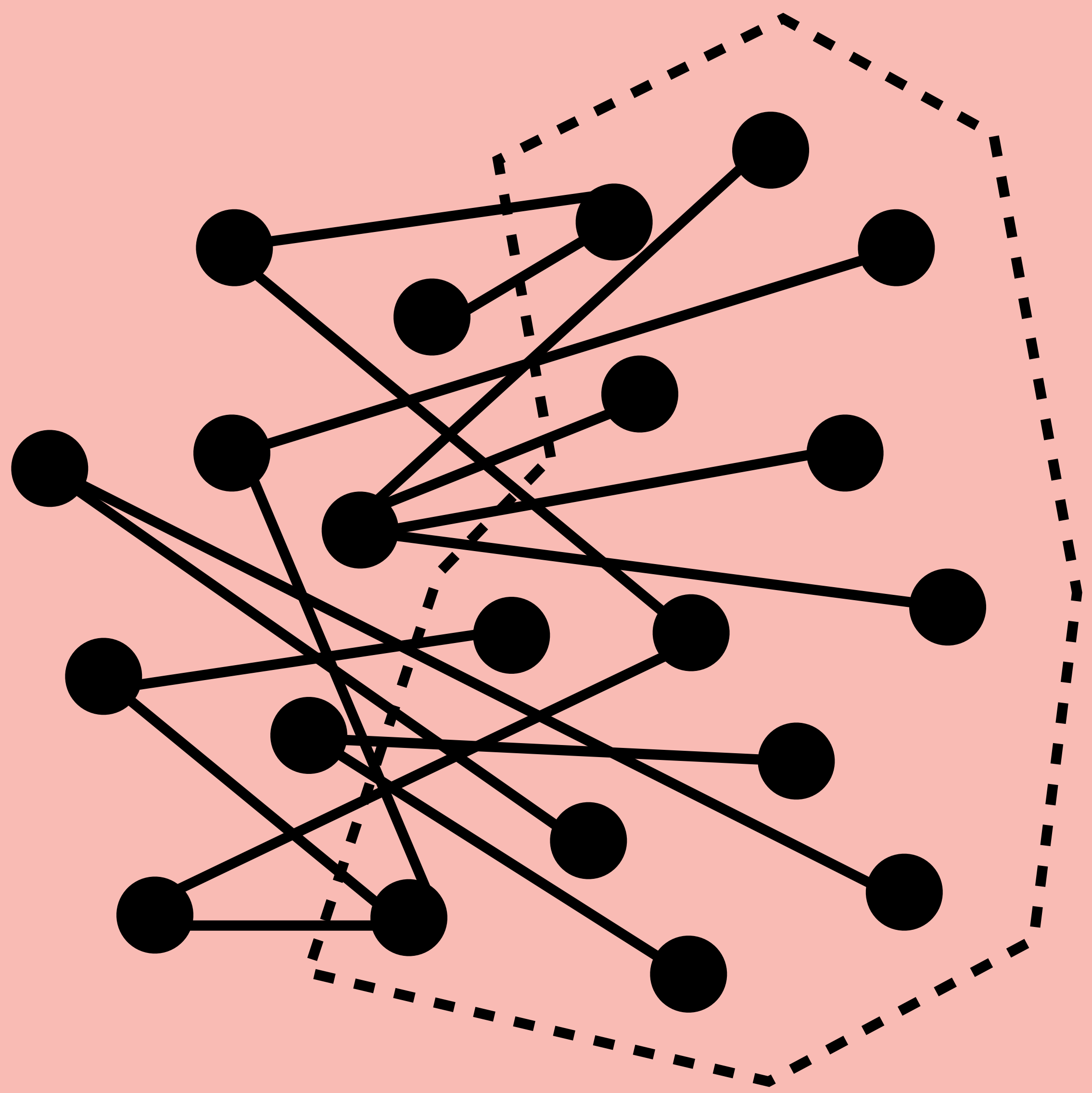

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How to Use DBHP? A Graph View

- Think of each row of M_t as a random edge and w_t picks the edges.

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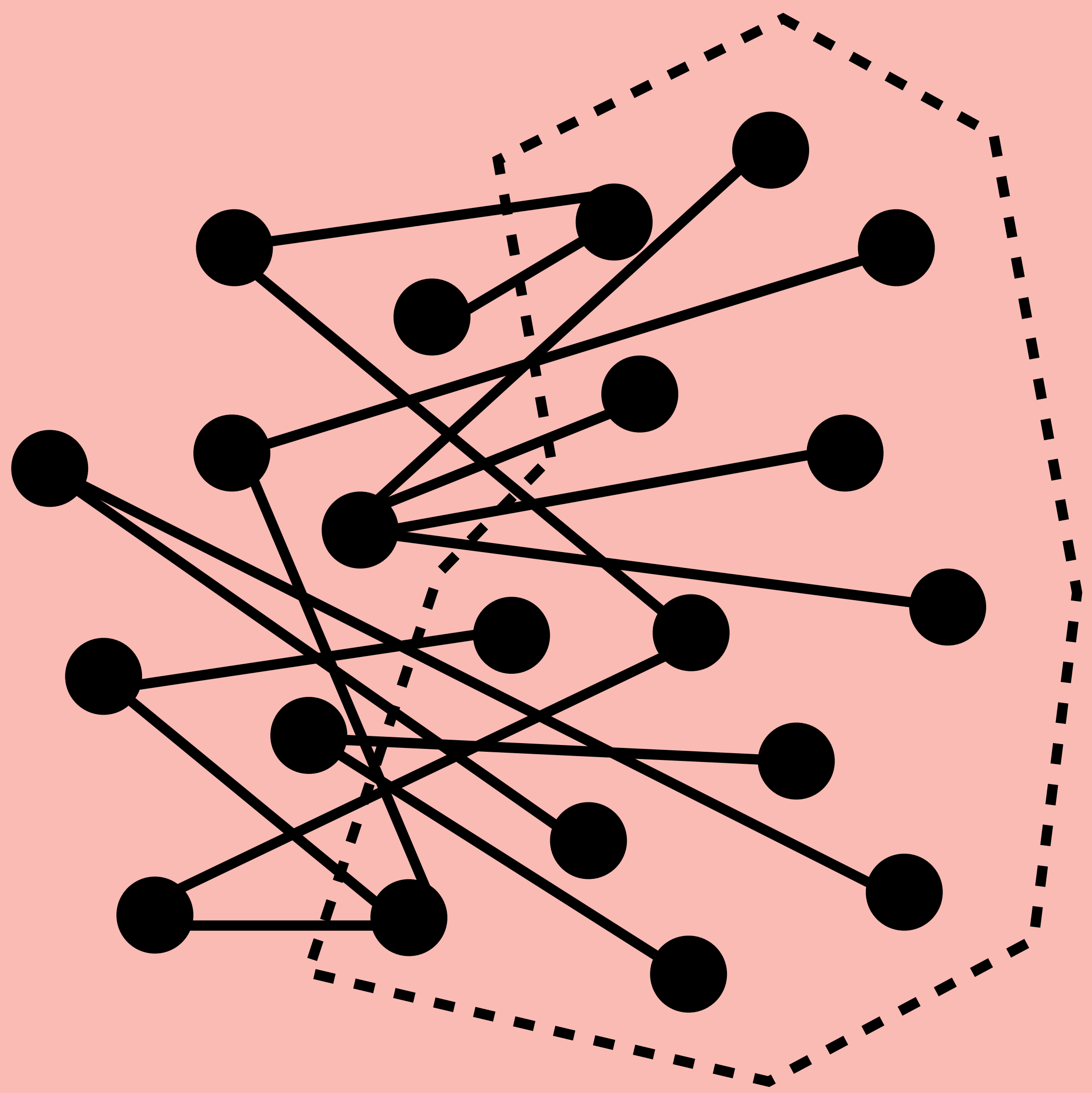

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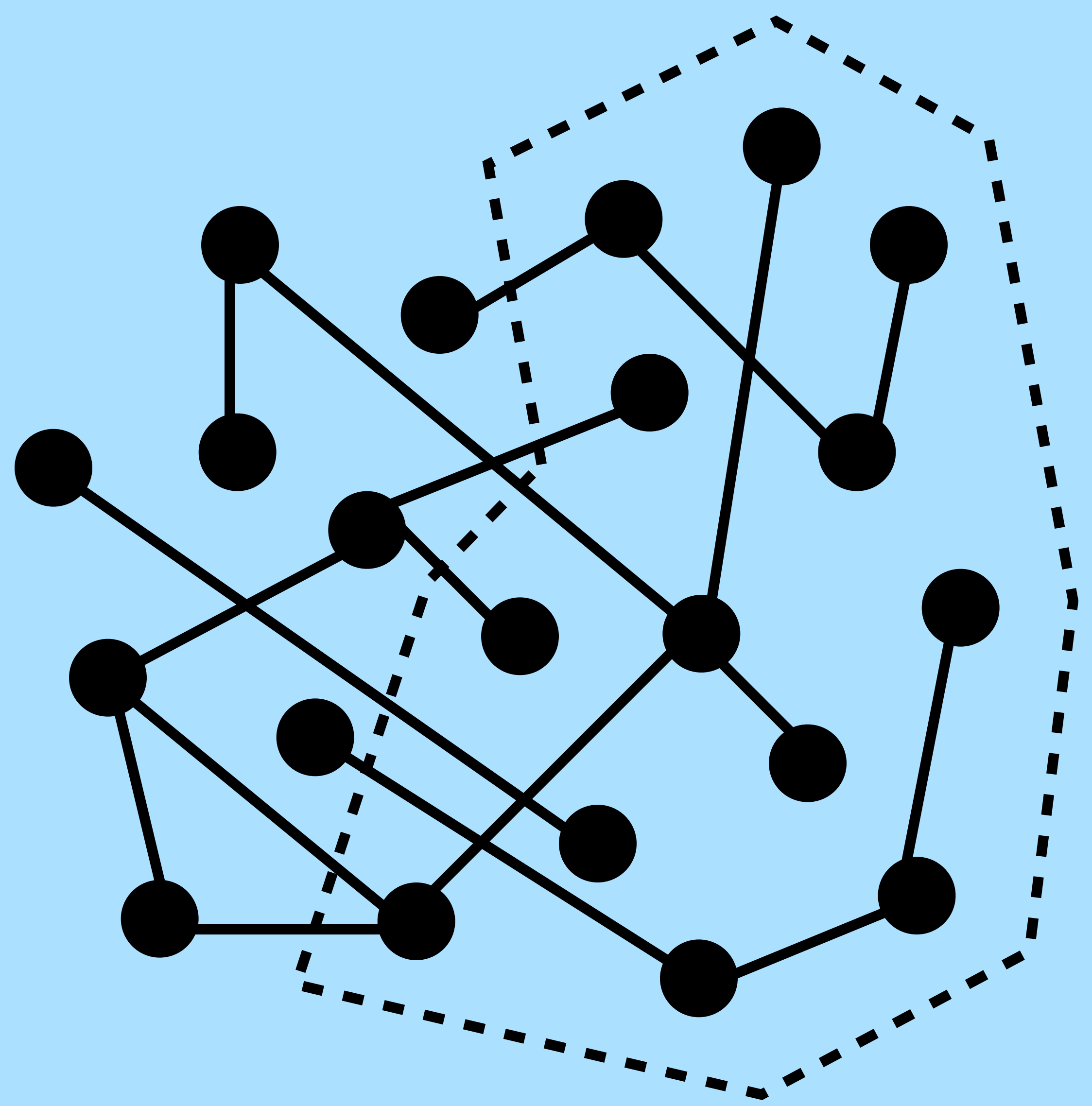

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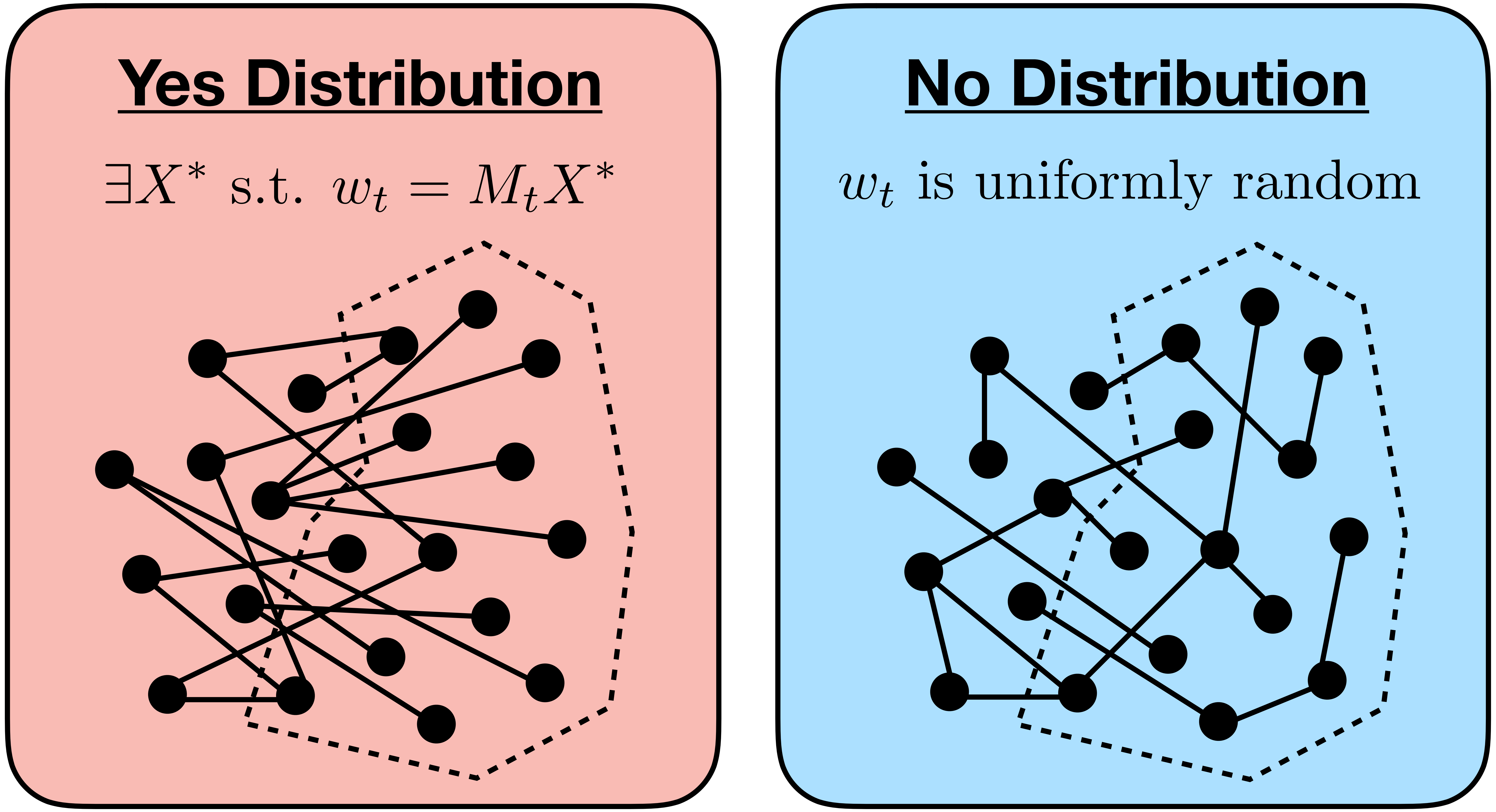

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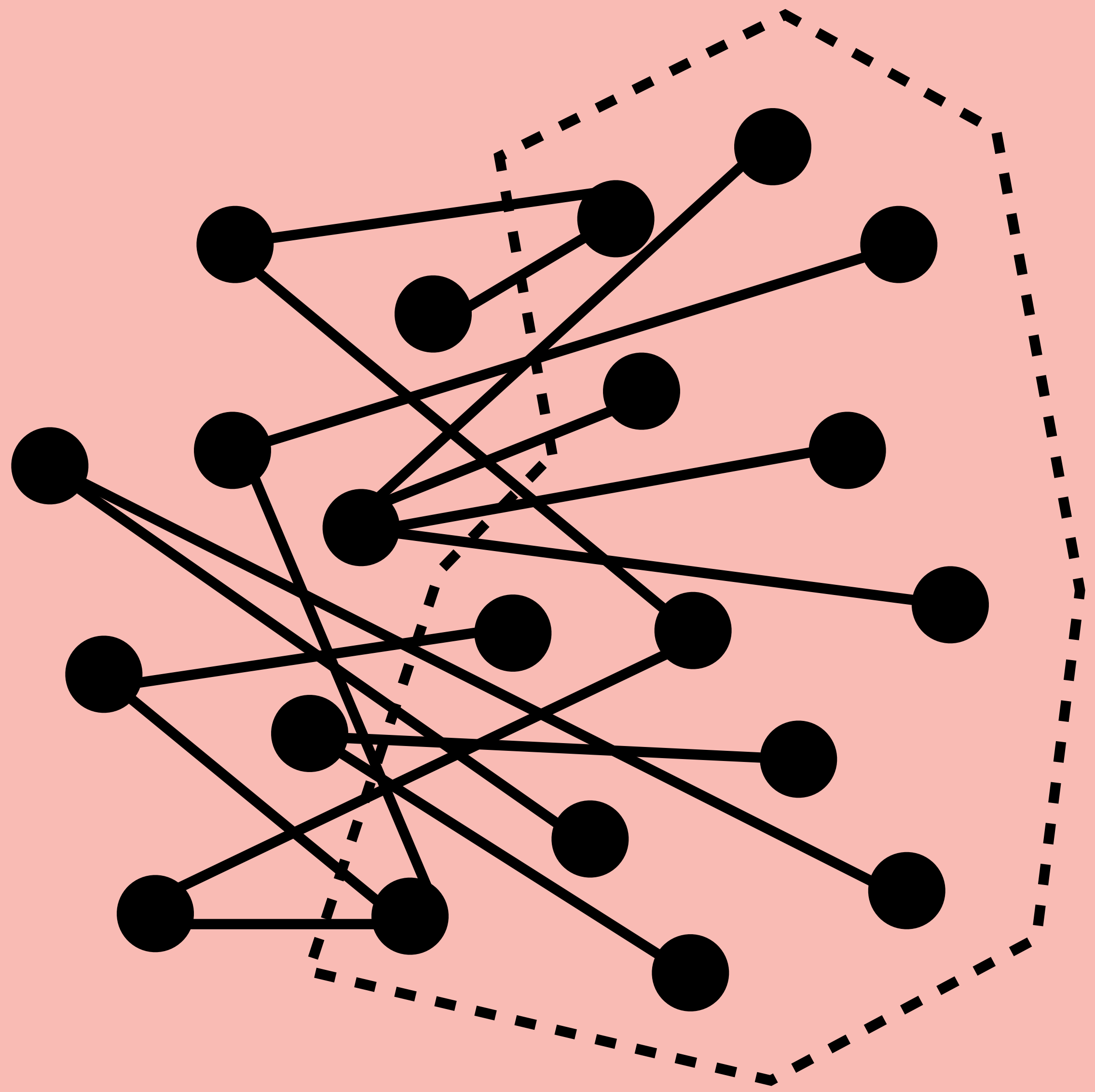

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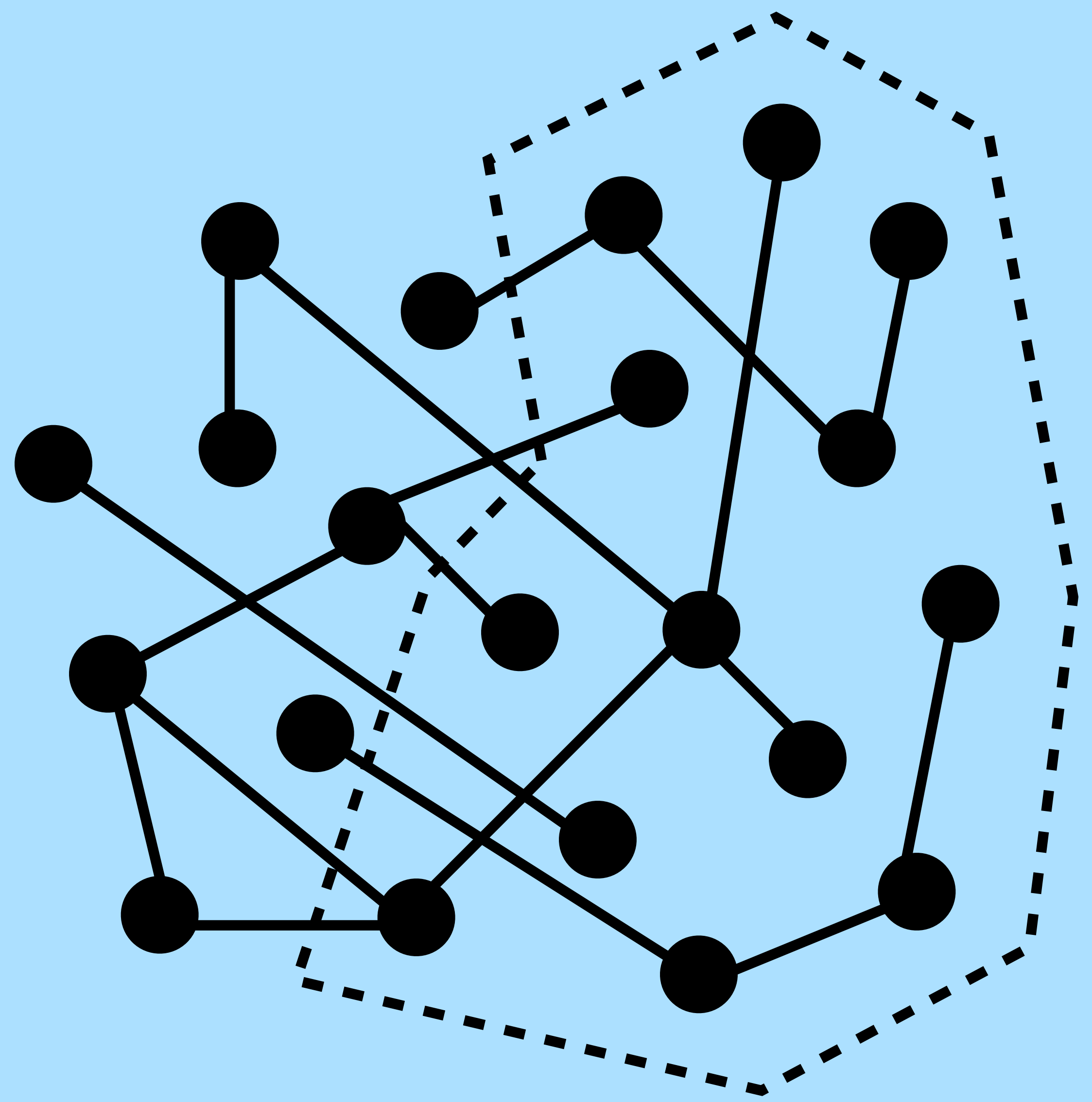

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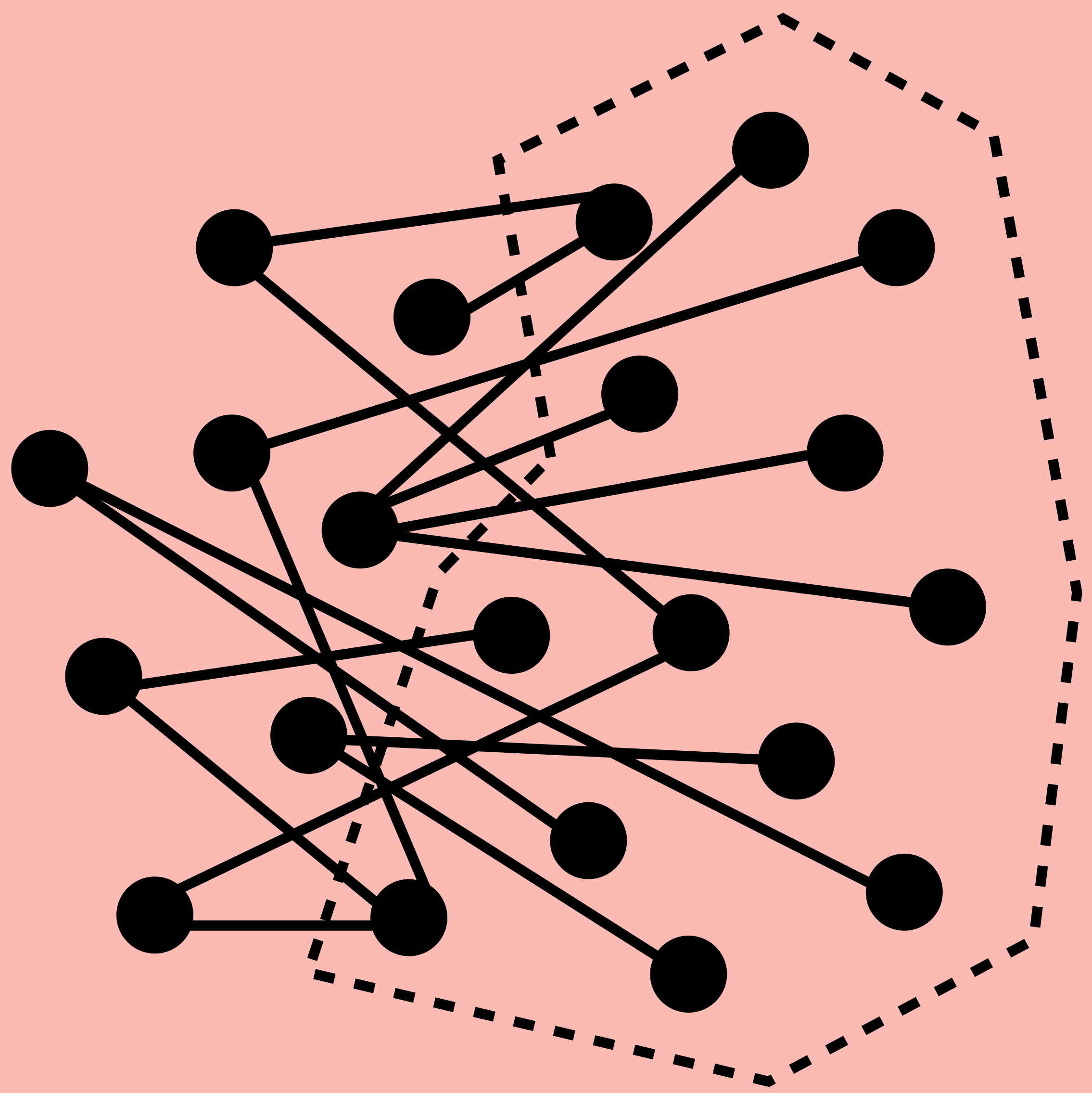

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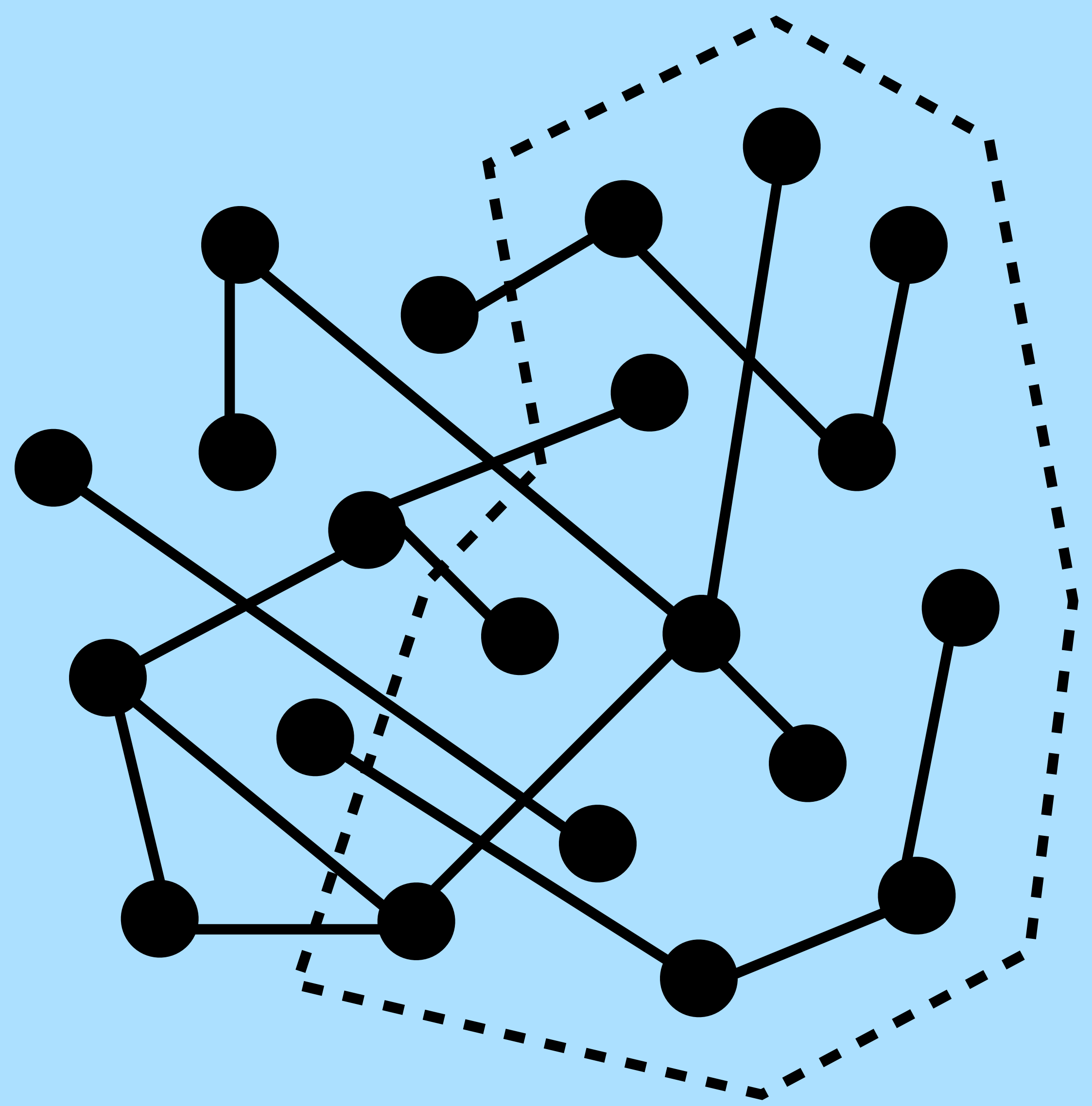

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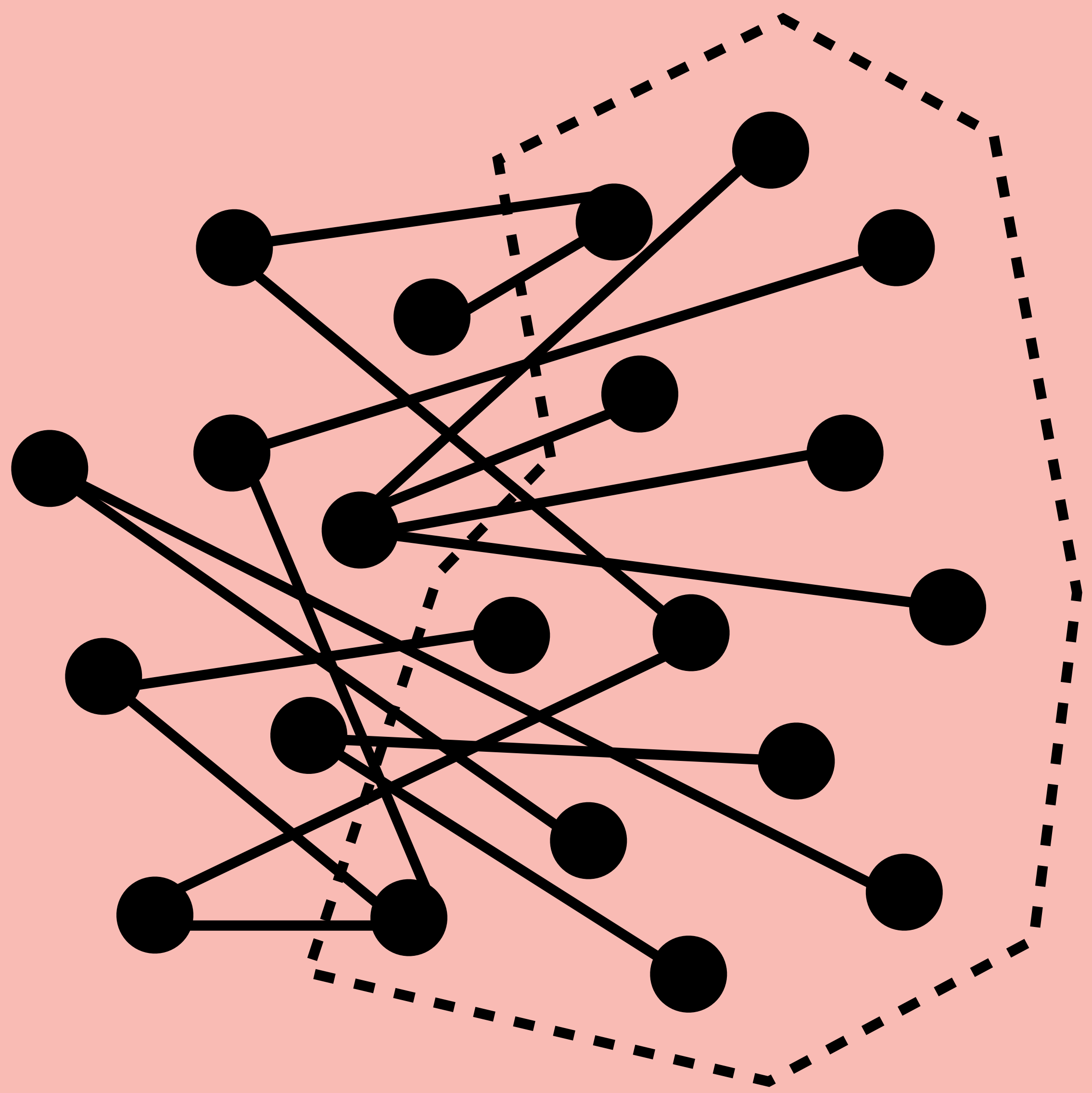

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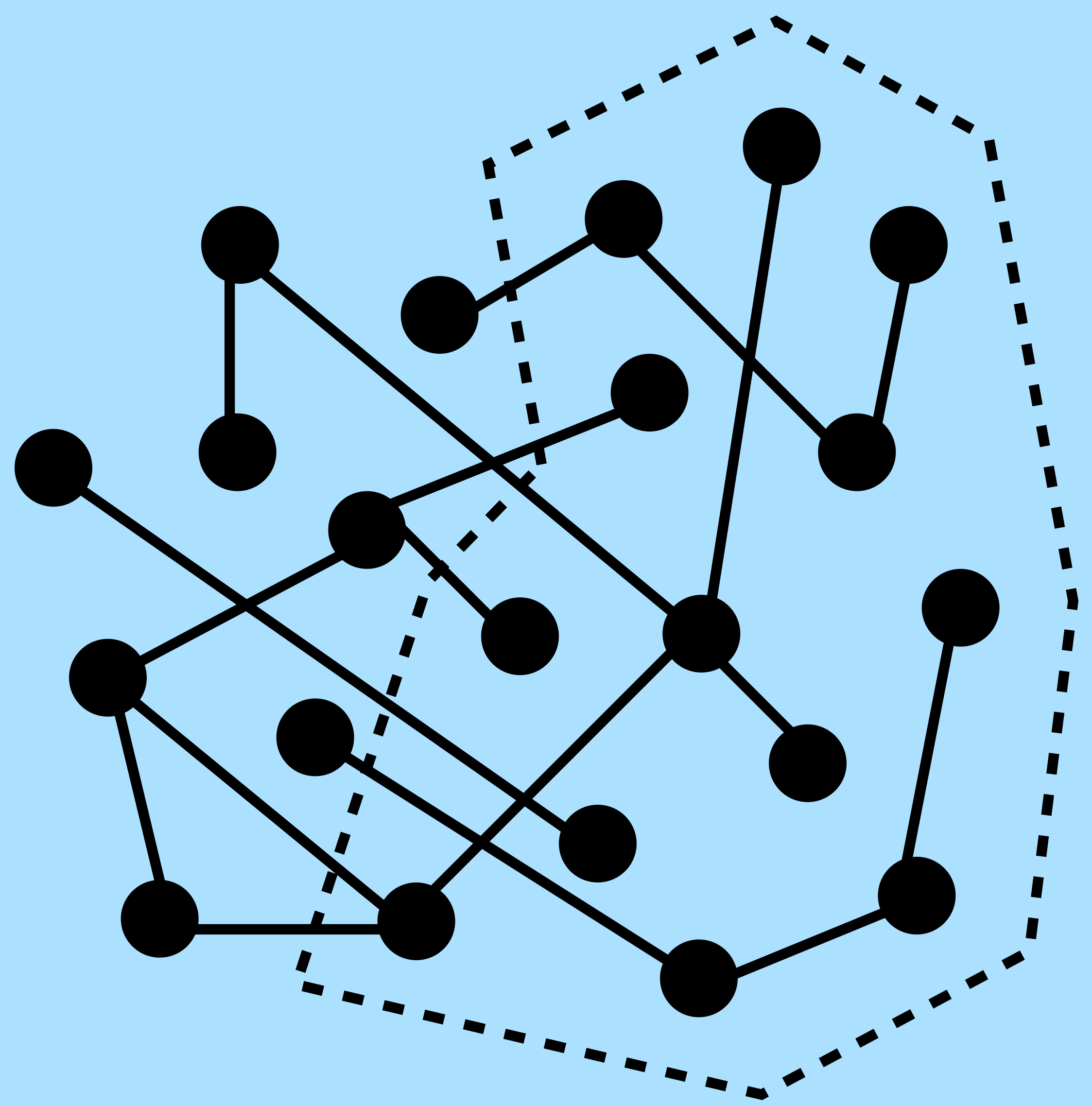

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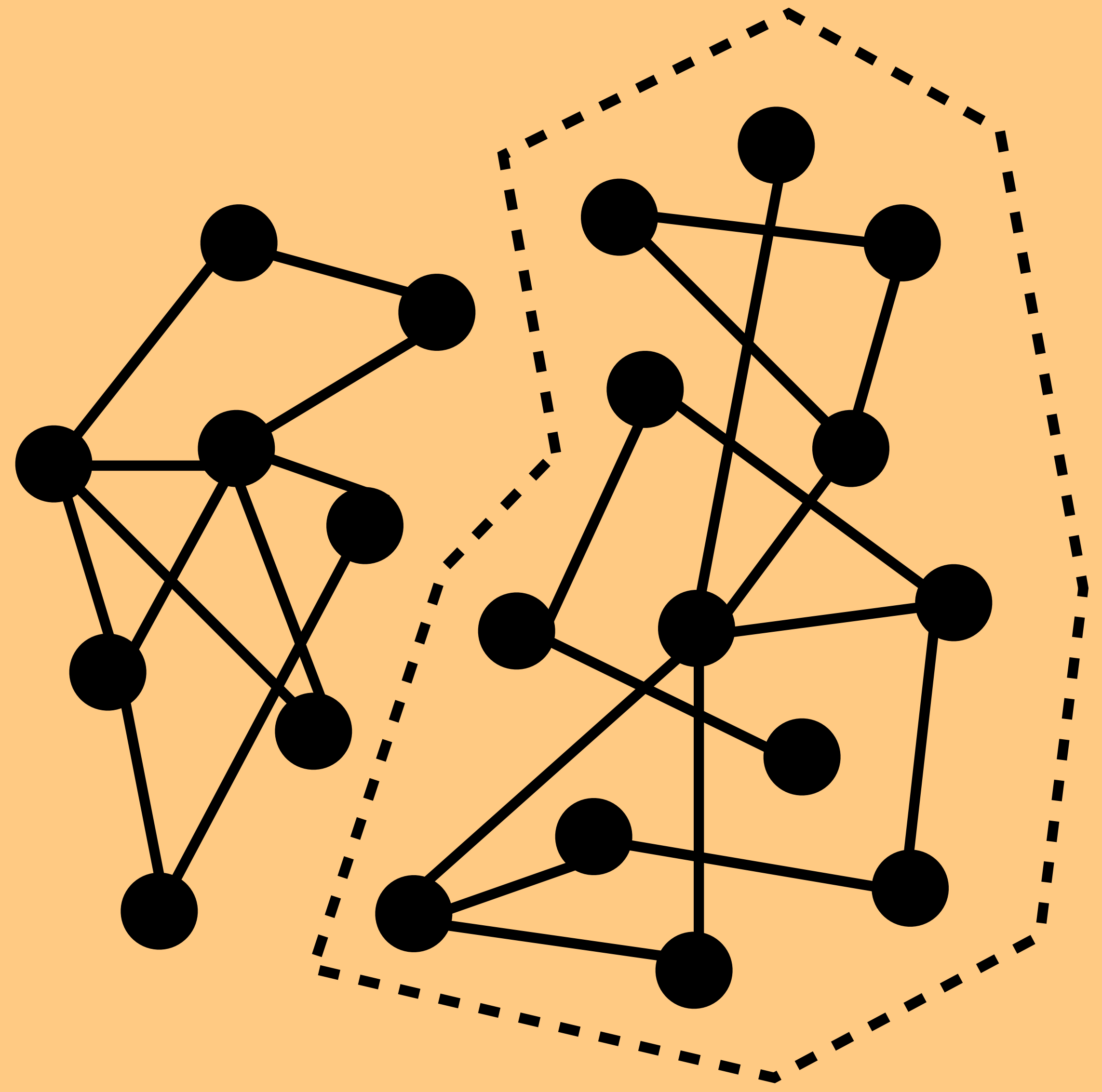

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Reducing DBHP to Max-CUT

Bob 1

$w_1 \in \{0,1\}^{0.01n}$

Bob 2

$w_2 \in \{0,1\}^{0.01n}$

…

Bob *T*

$w_T \in \{0,1\}^{0.01n}$

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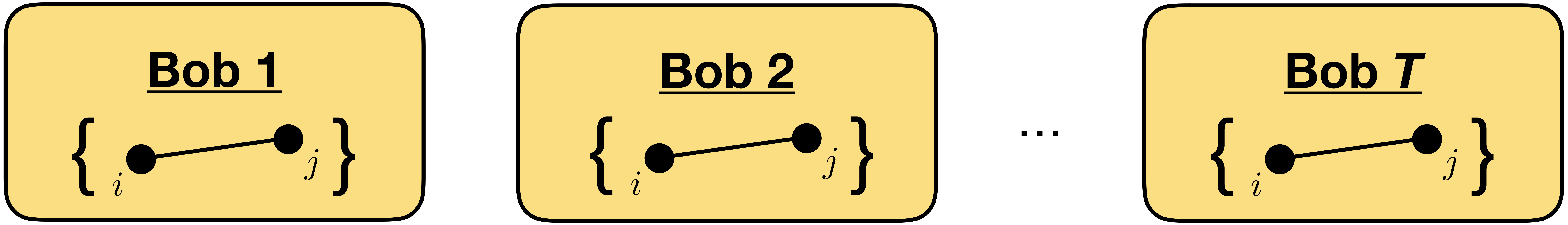

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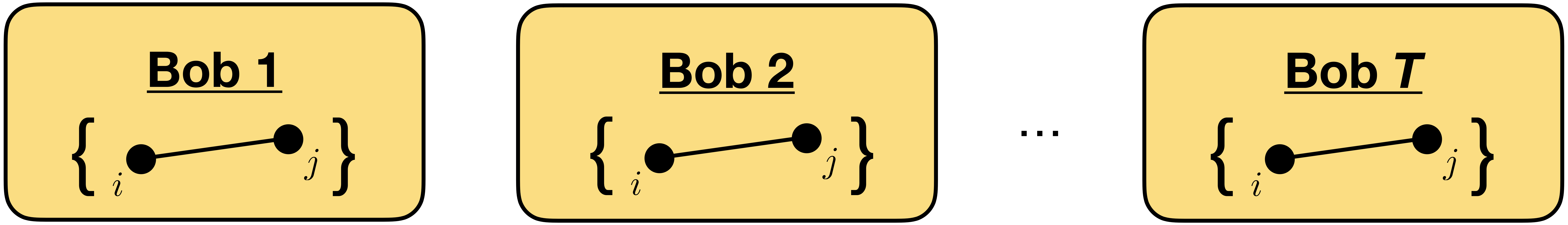

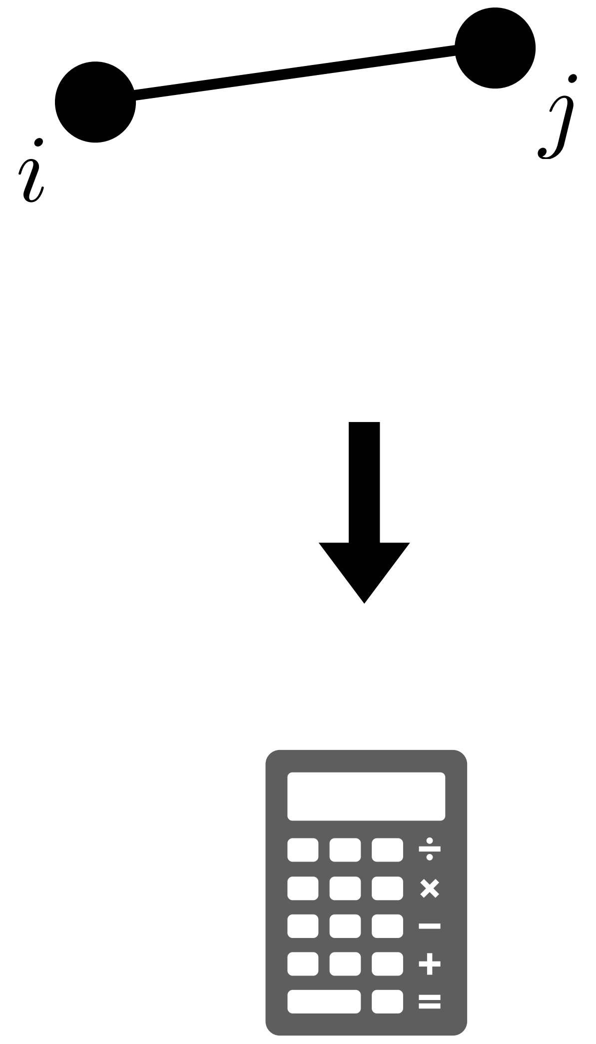

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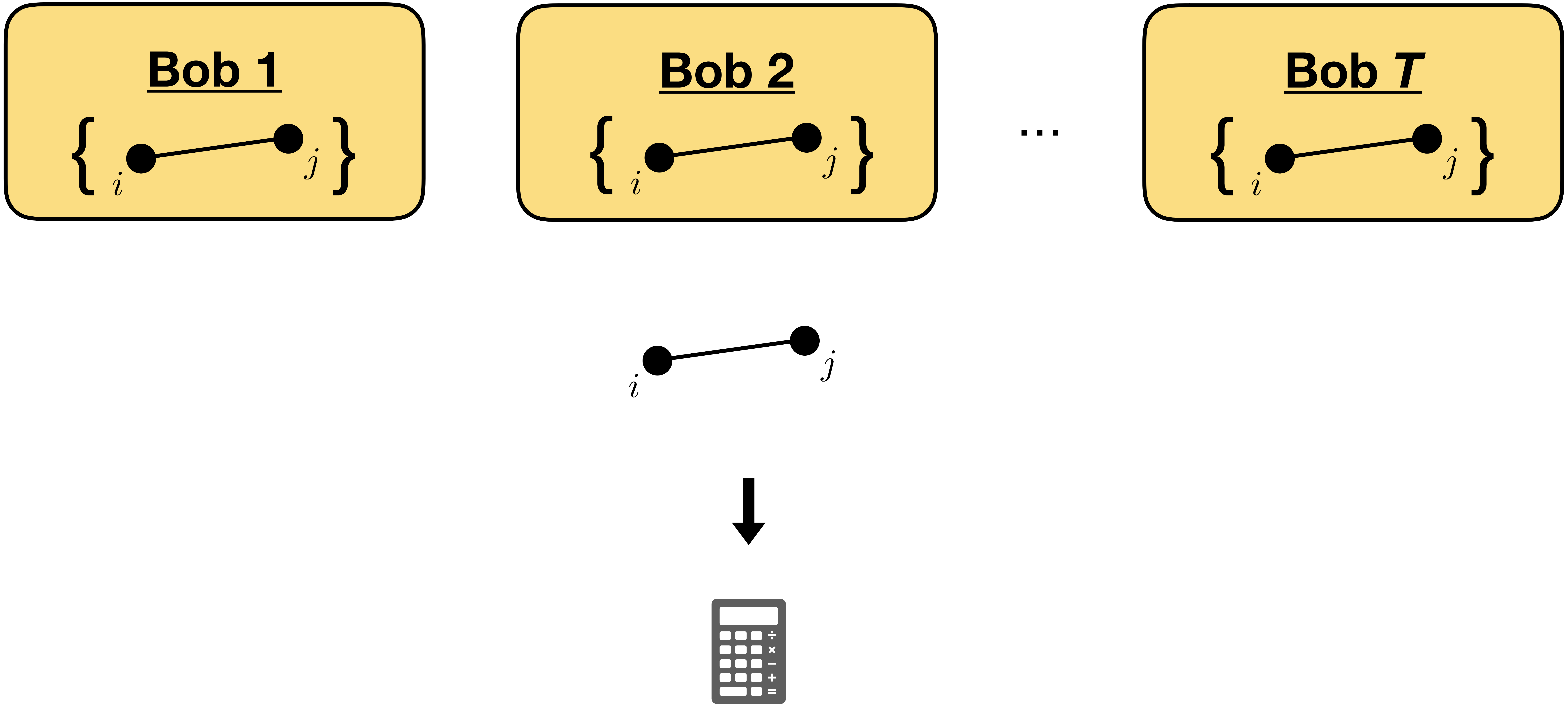

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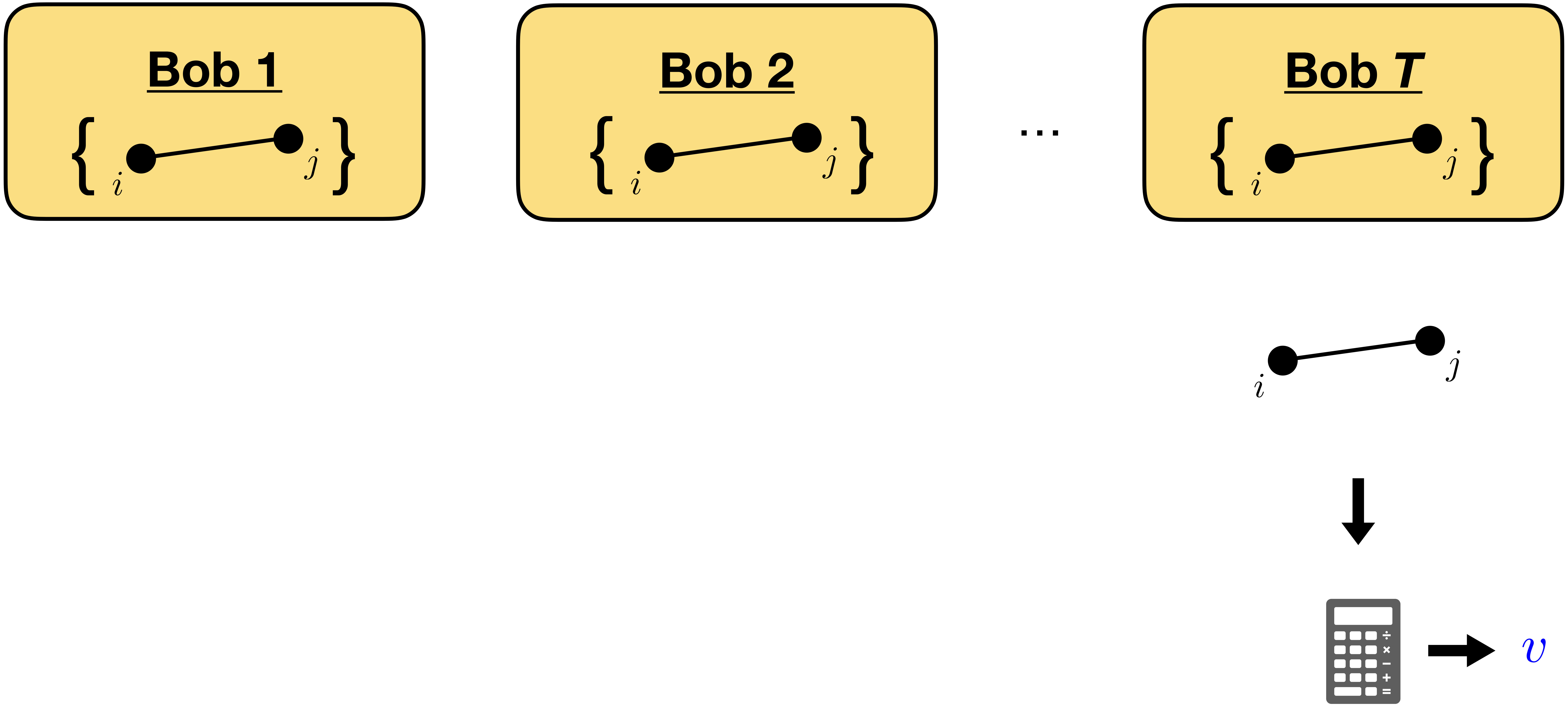

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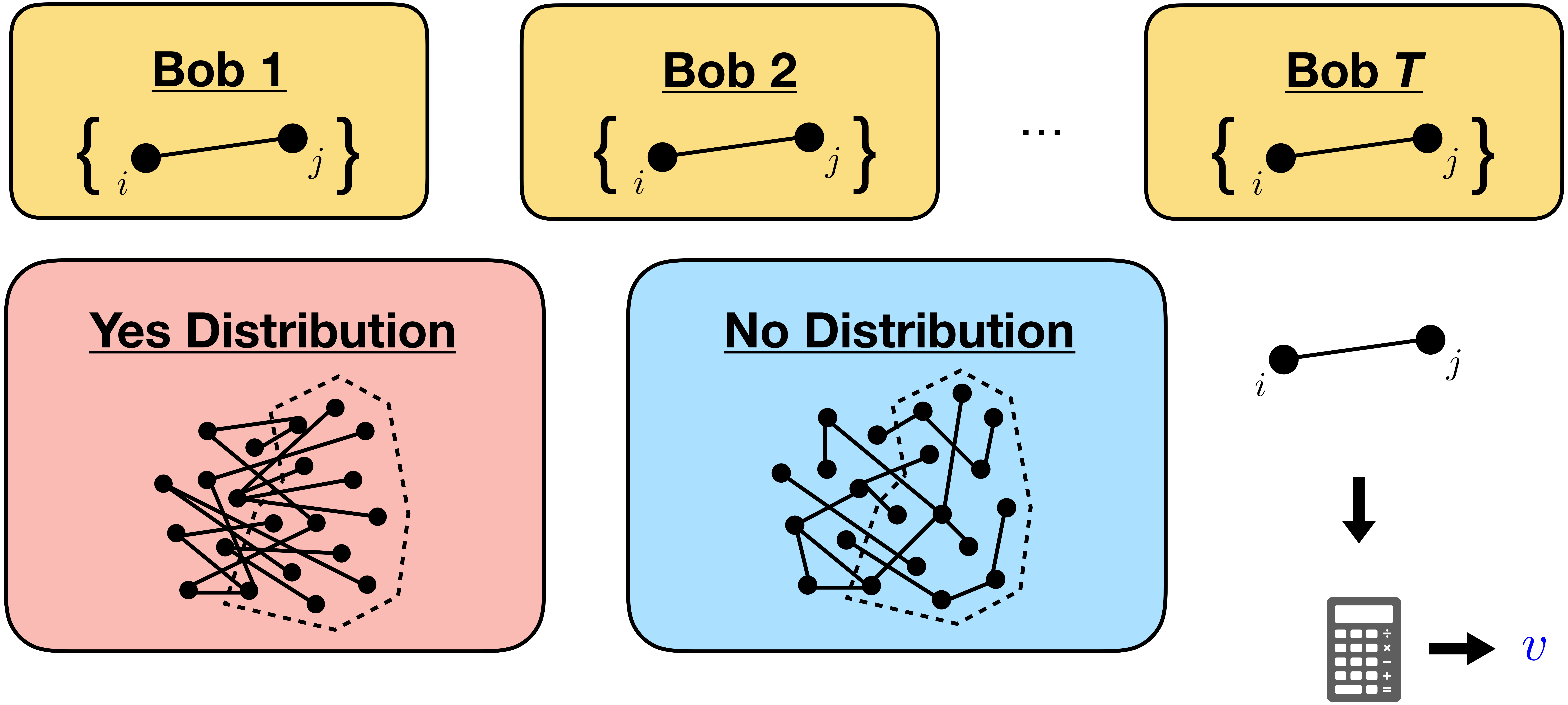

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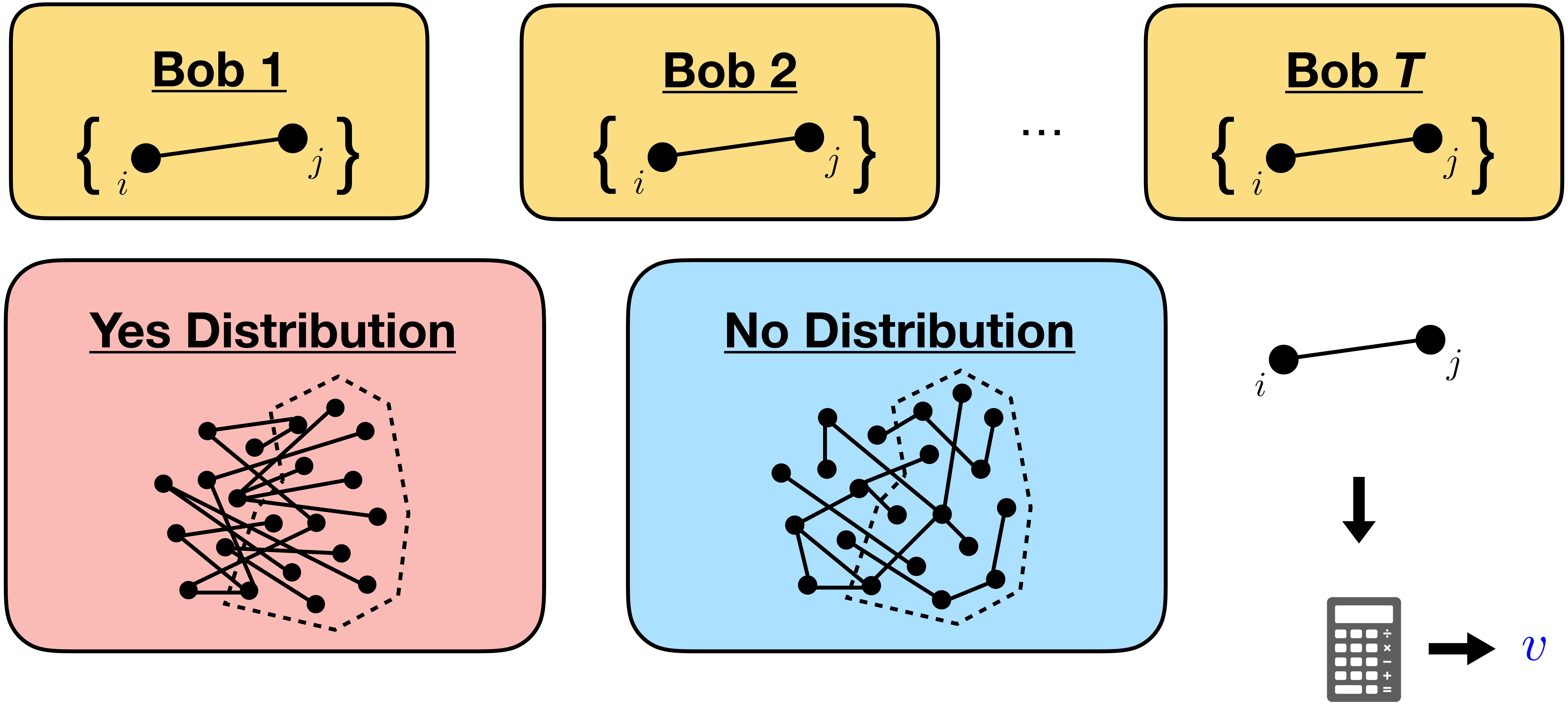

$\mathrm{val}_{\mathcal{C}} = m$

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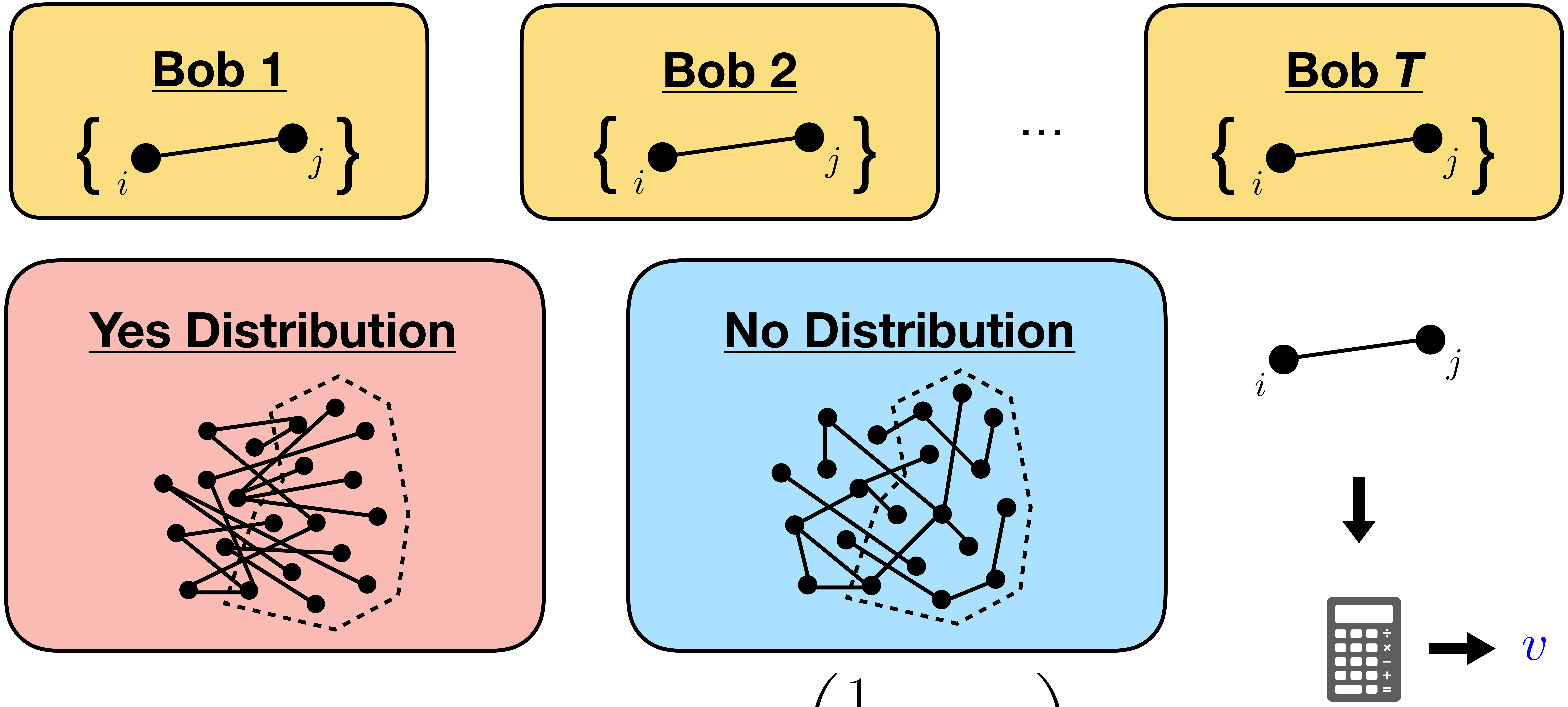

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$$\mathsf{val}_{\mathcal{C}} < \left(\frac{1}{2} + o(1)\right) \cdot m$$

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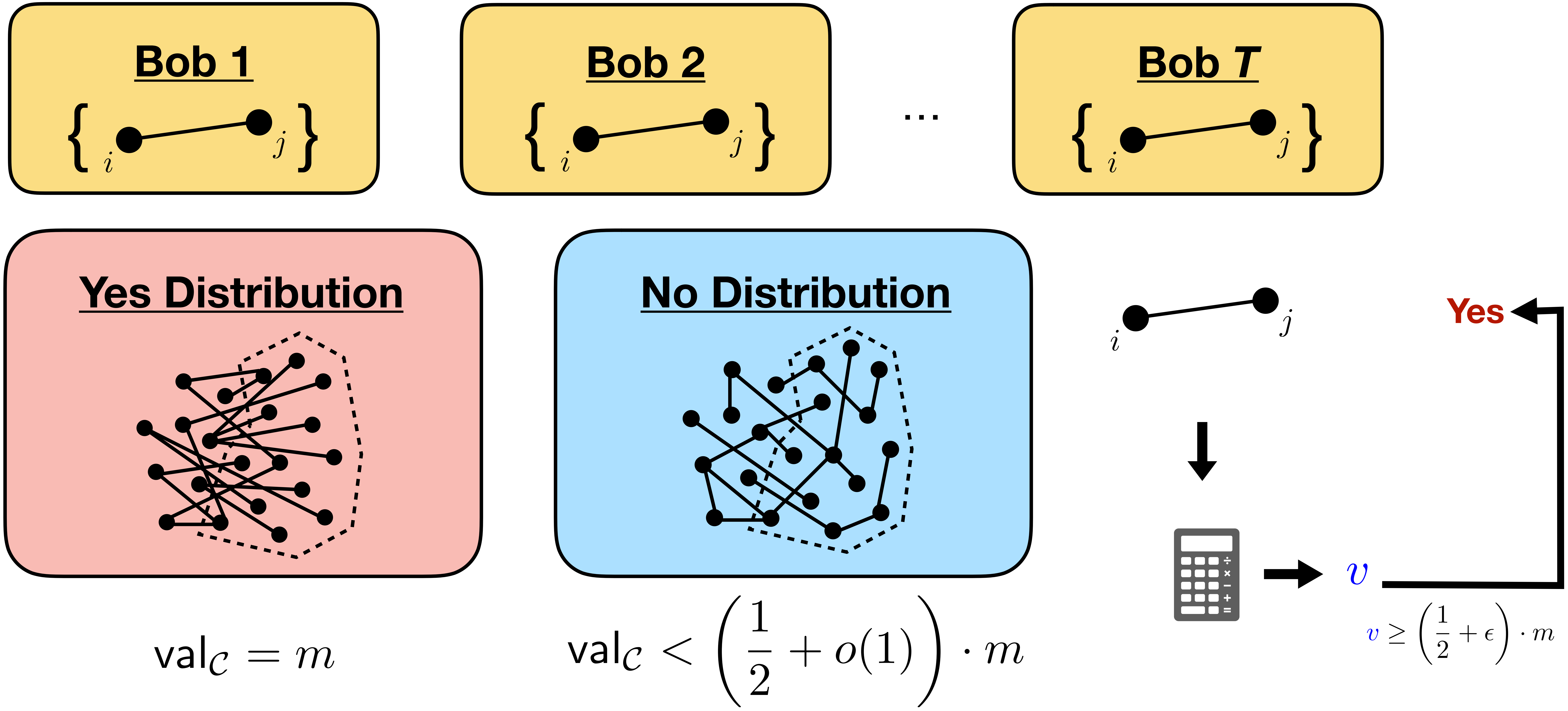

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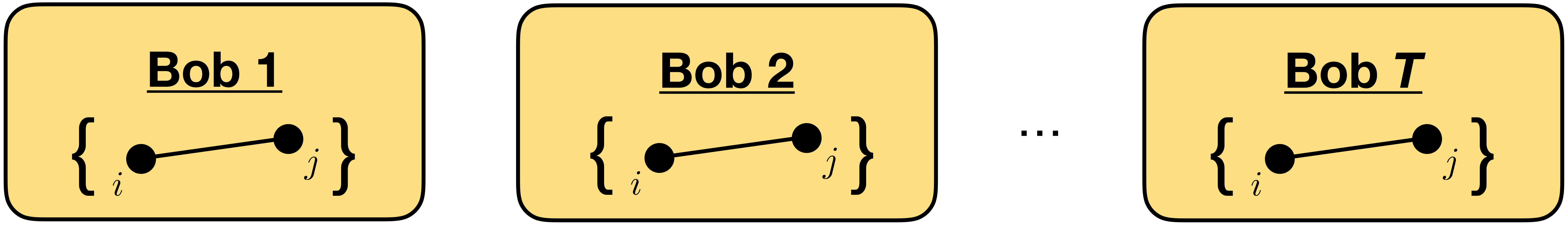

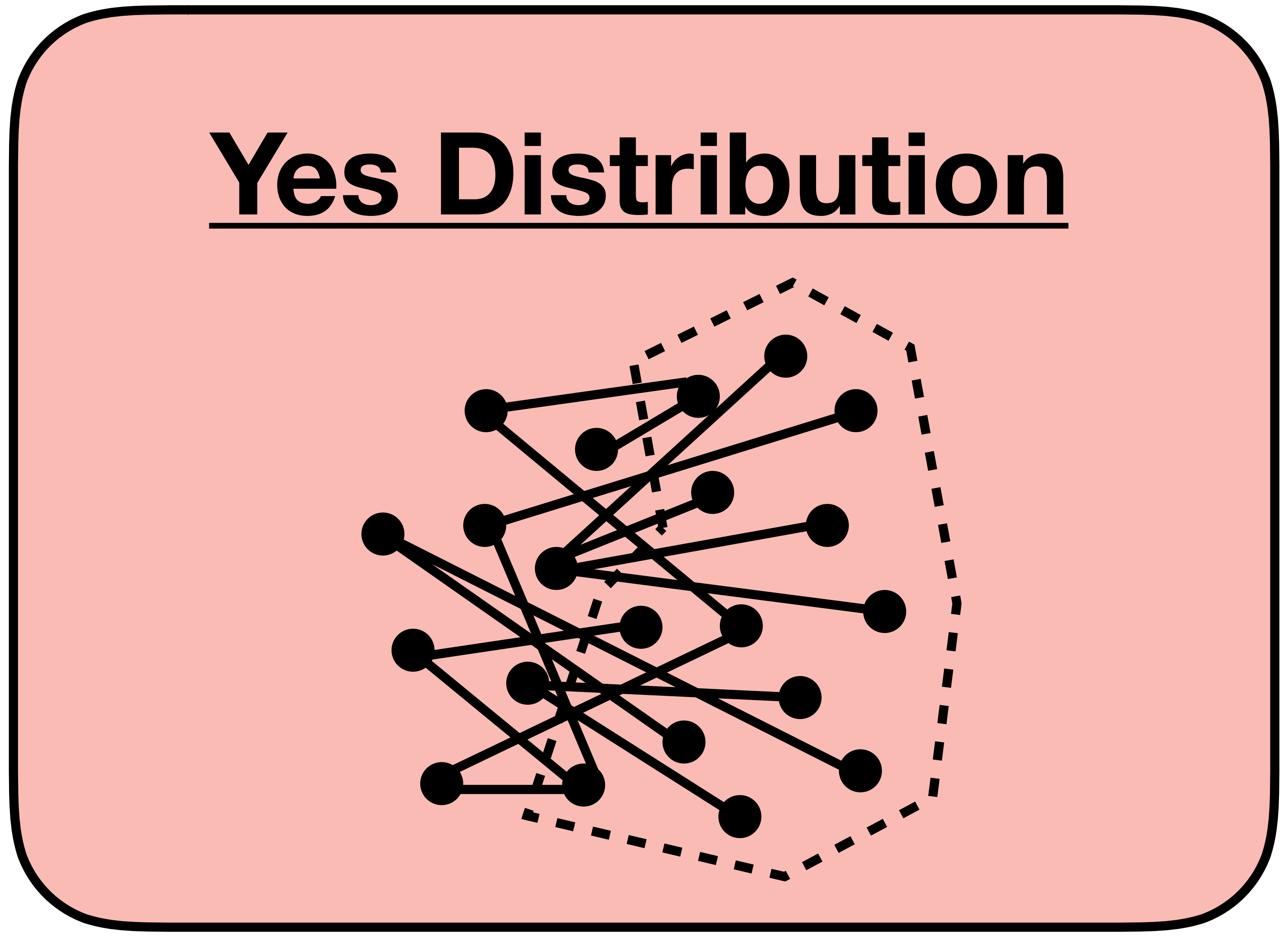

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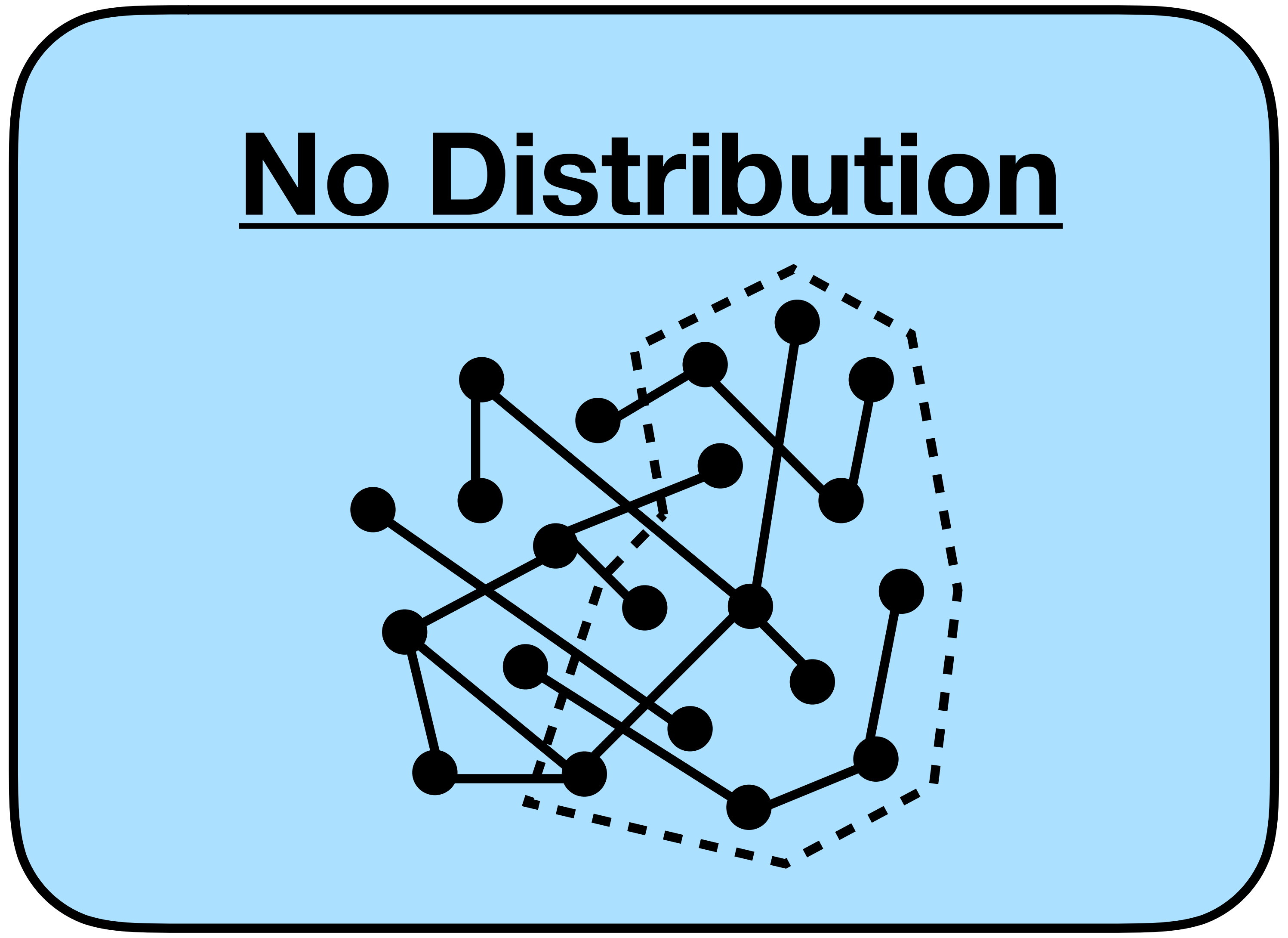

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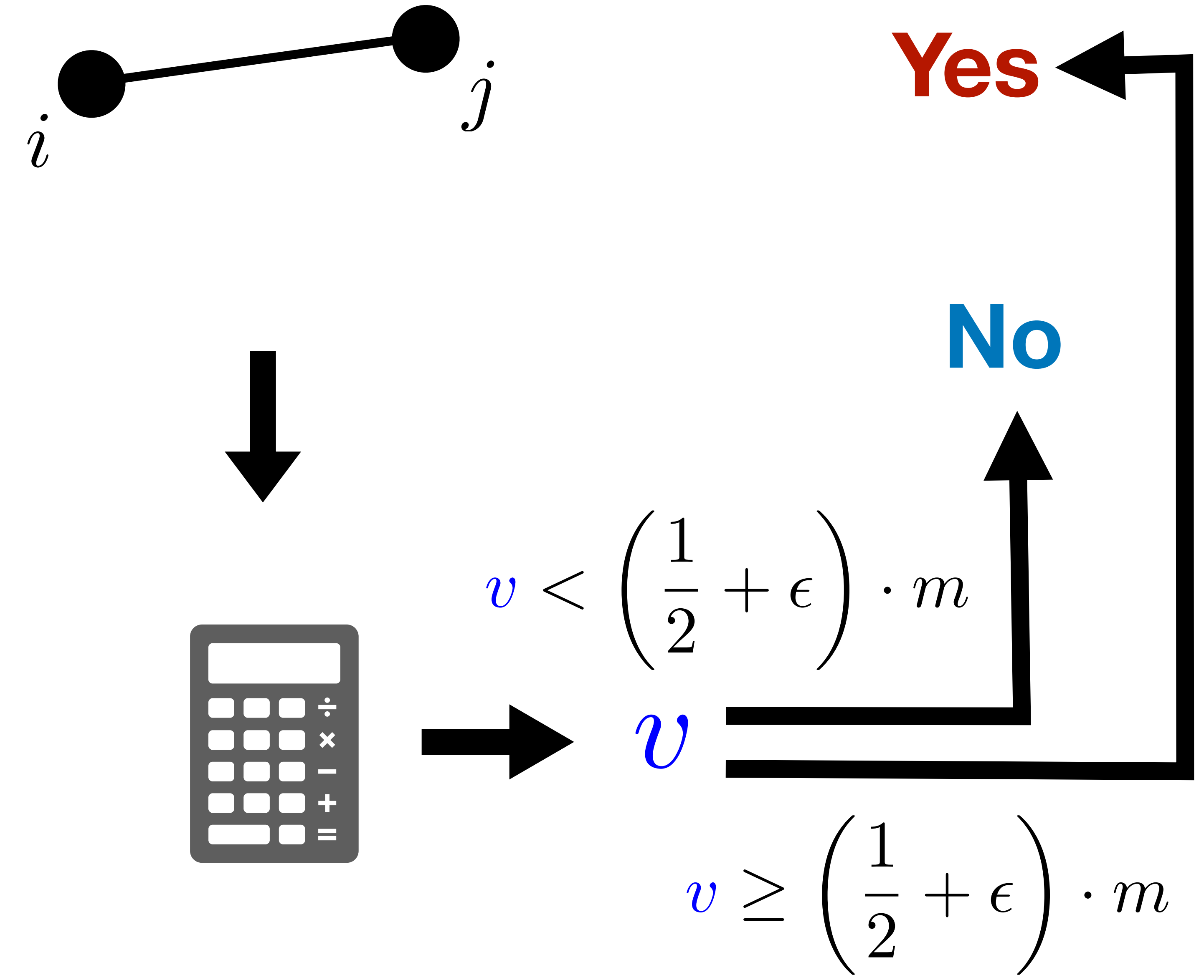

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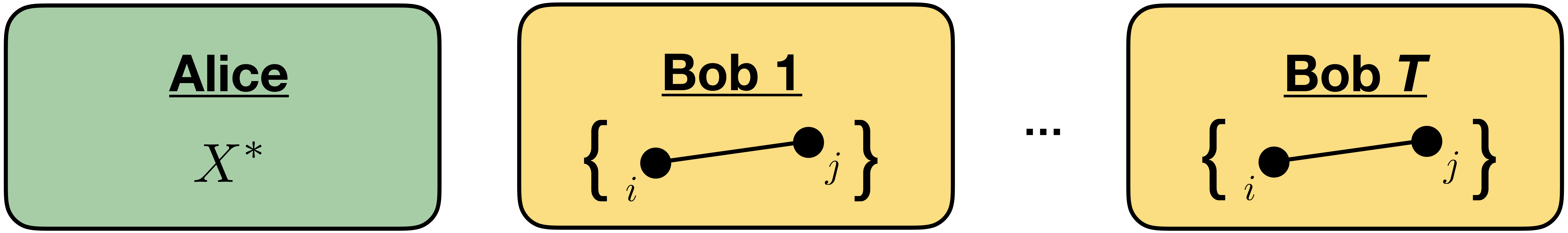
Reducing DBHP to Max-DICUT (Max-2AND)



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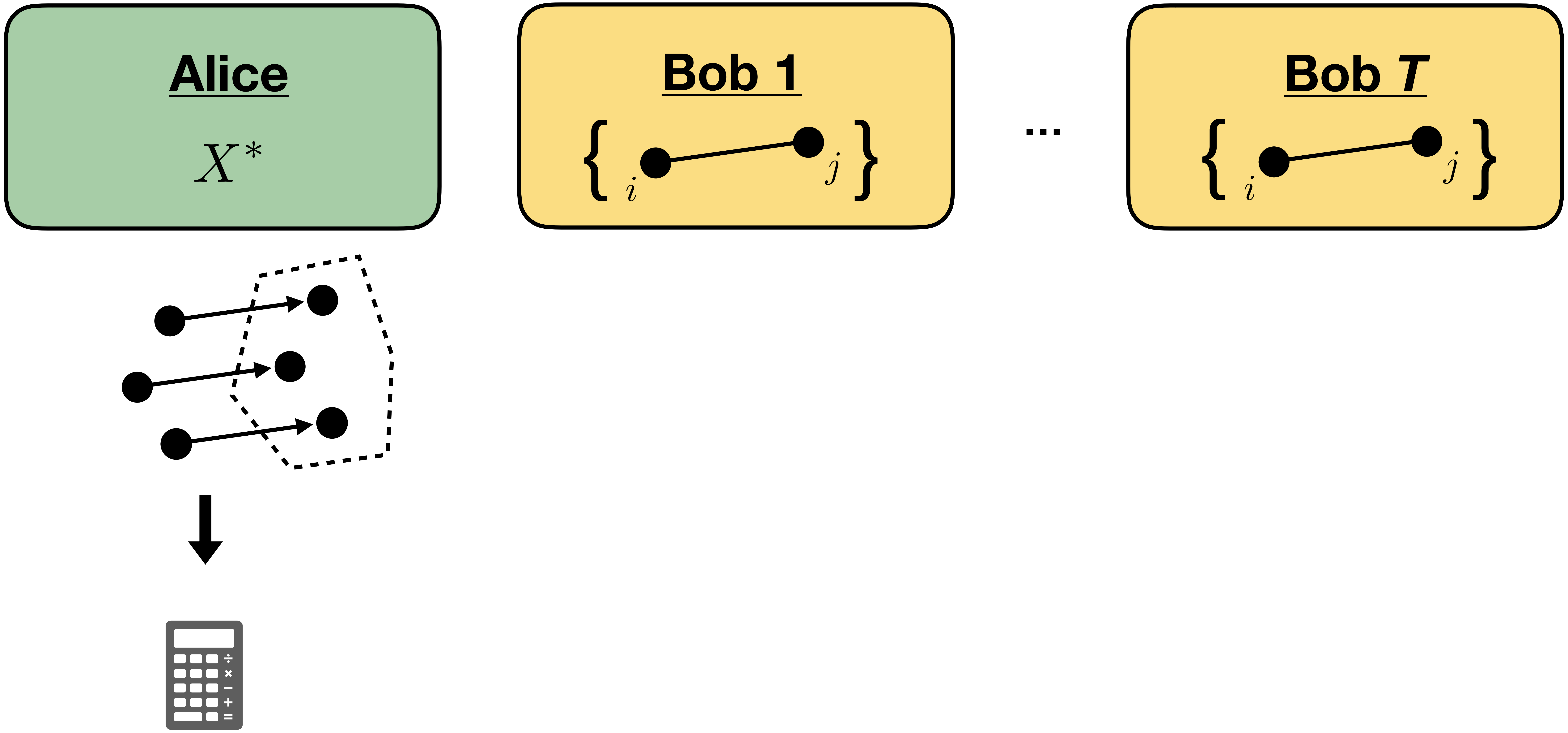

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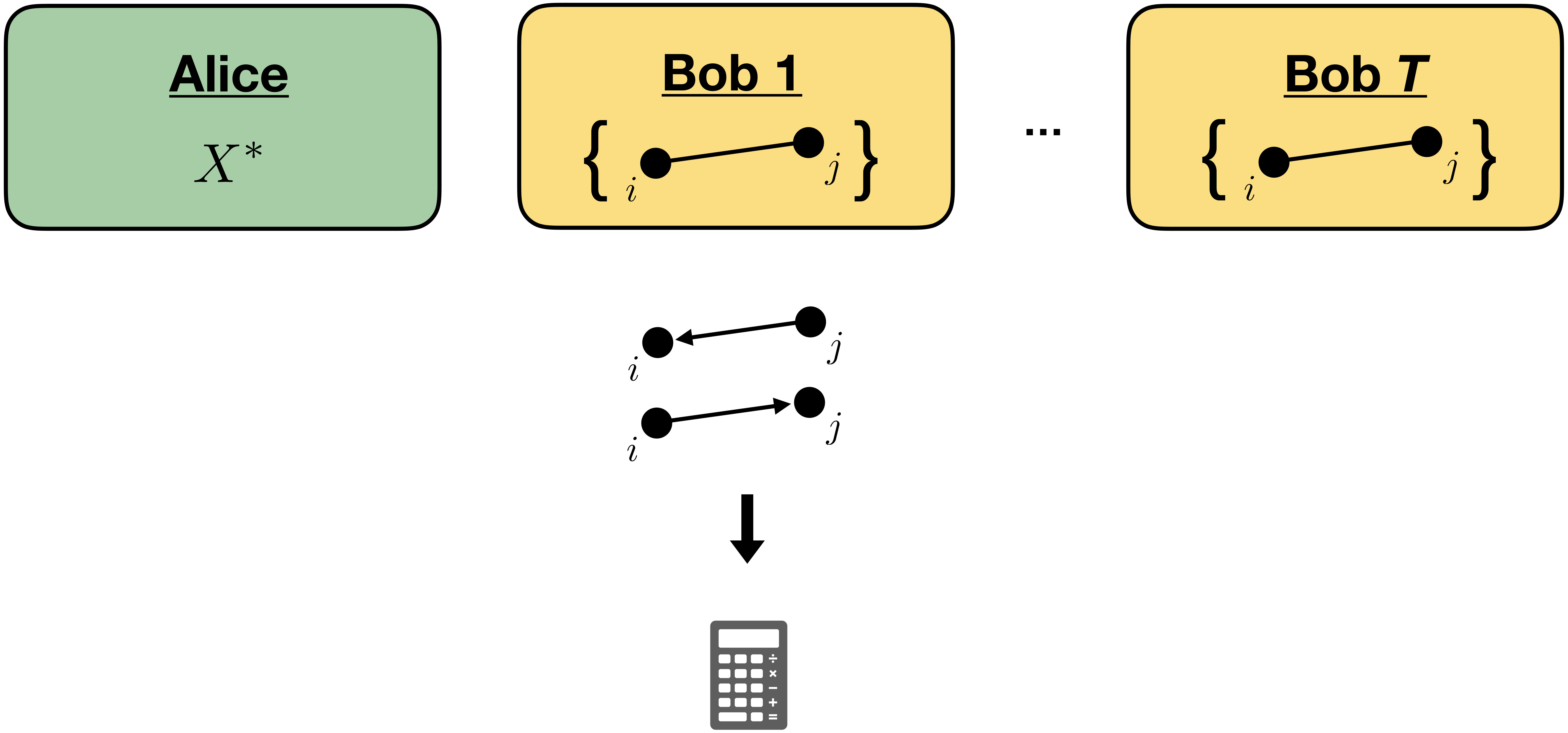

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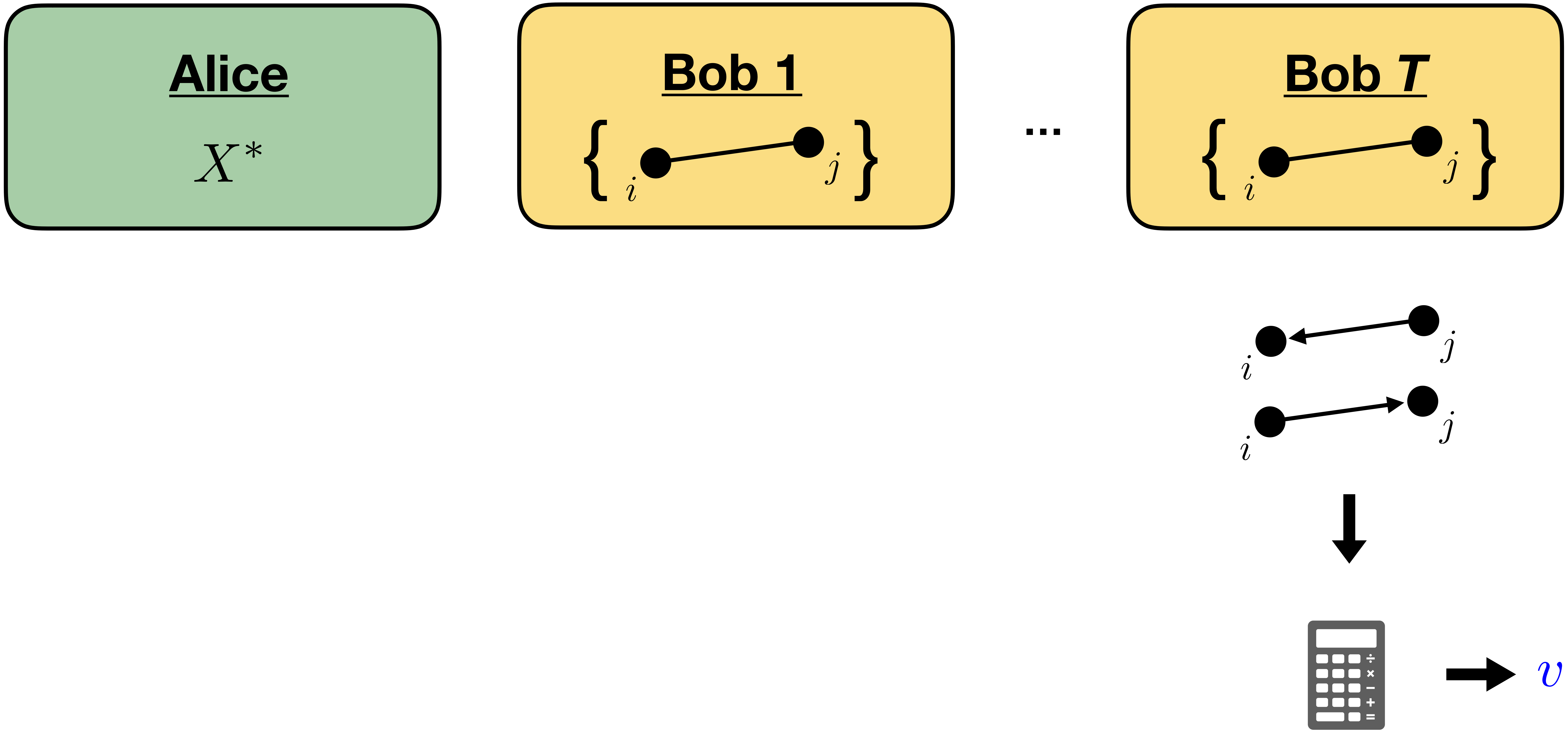

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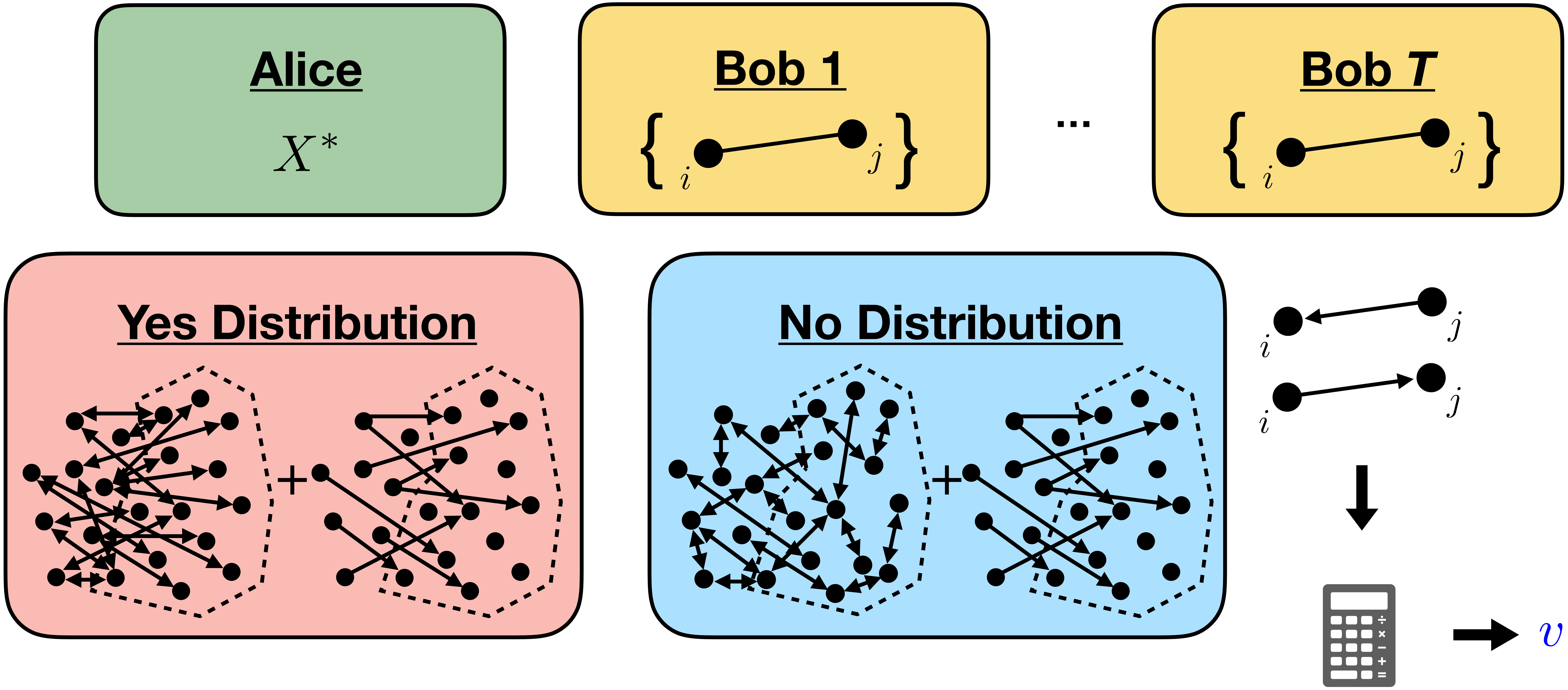

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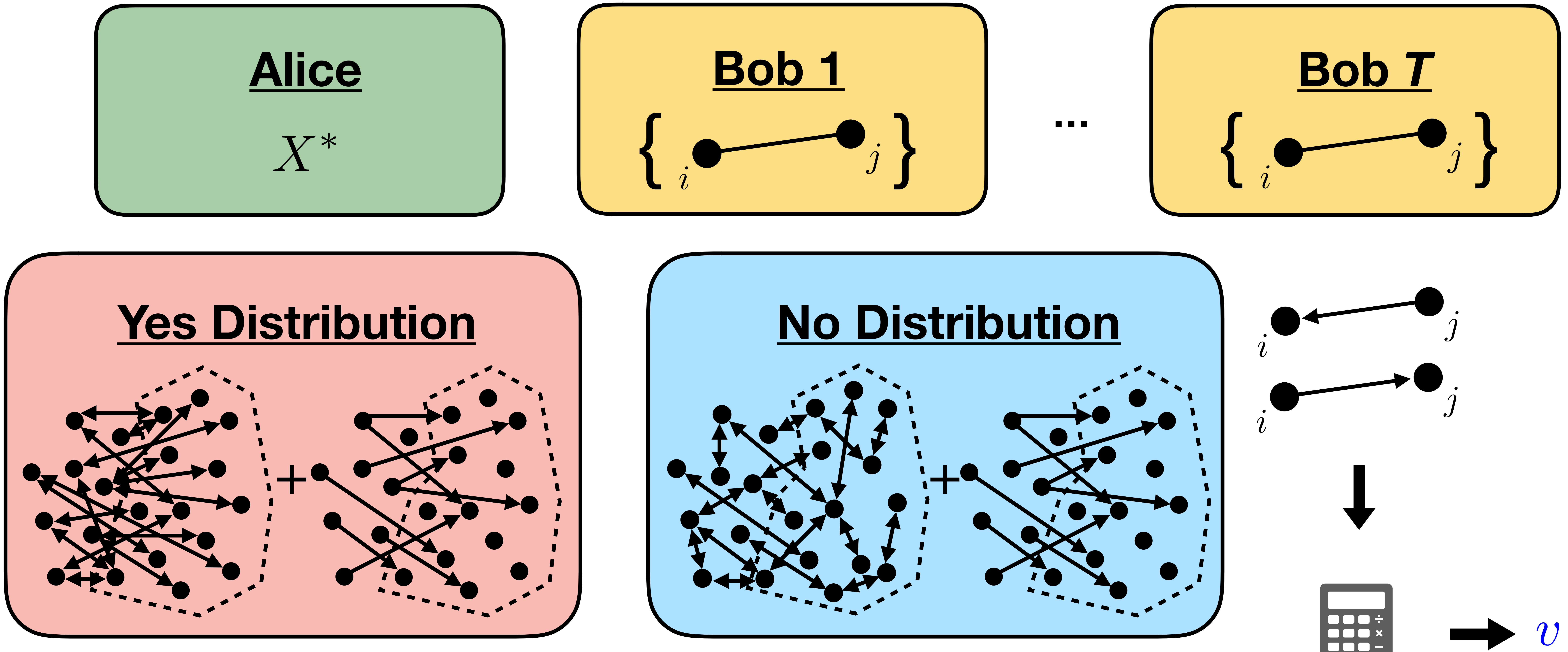

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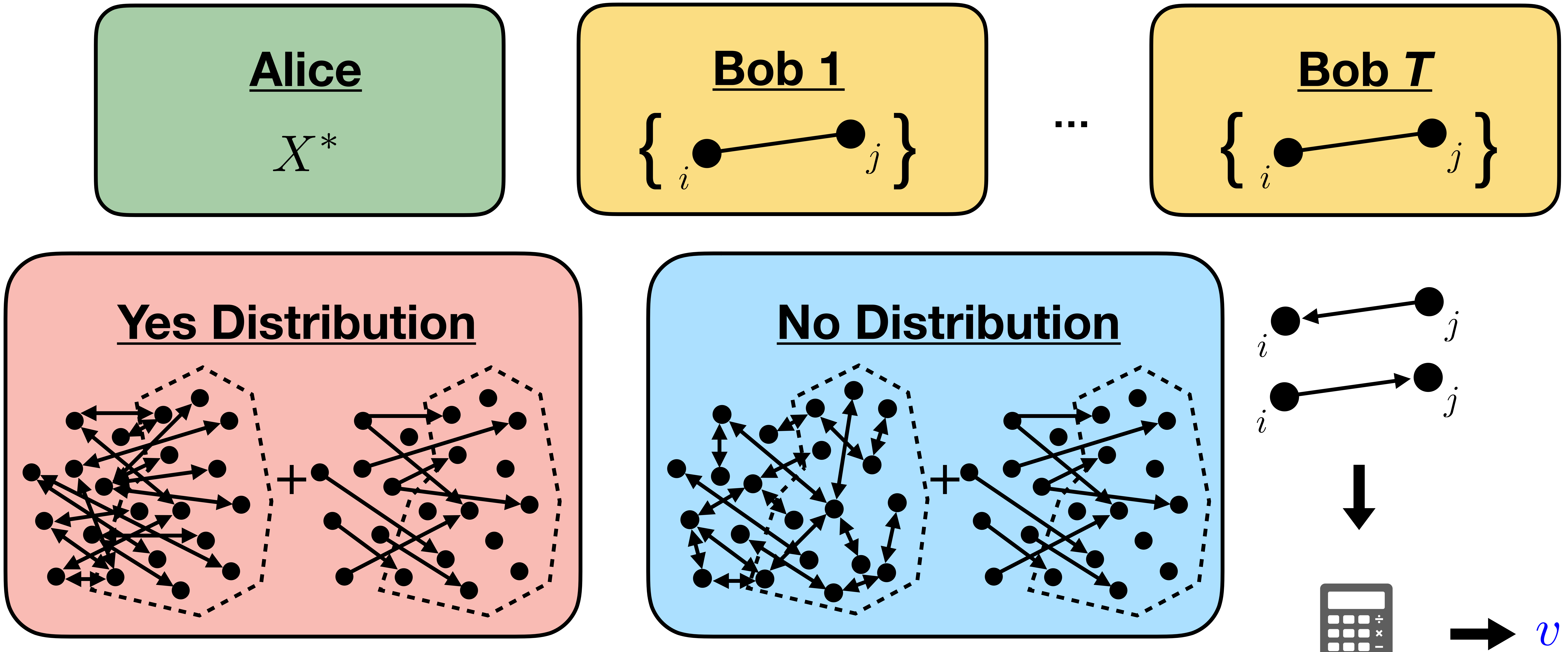

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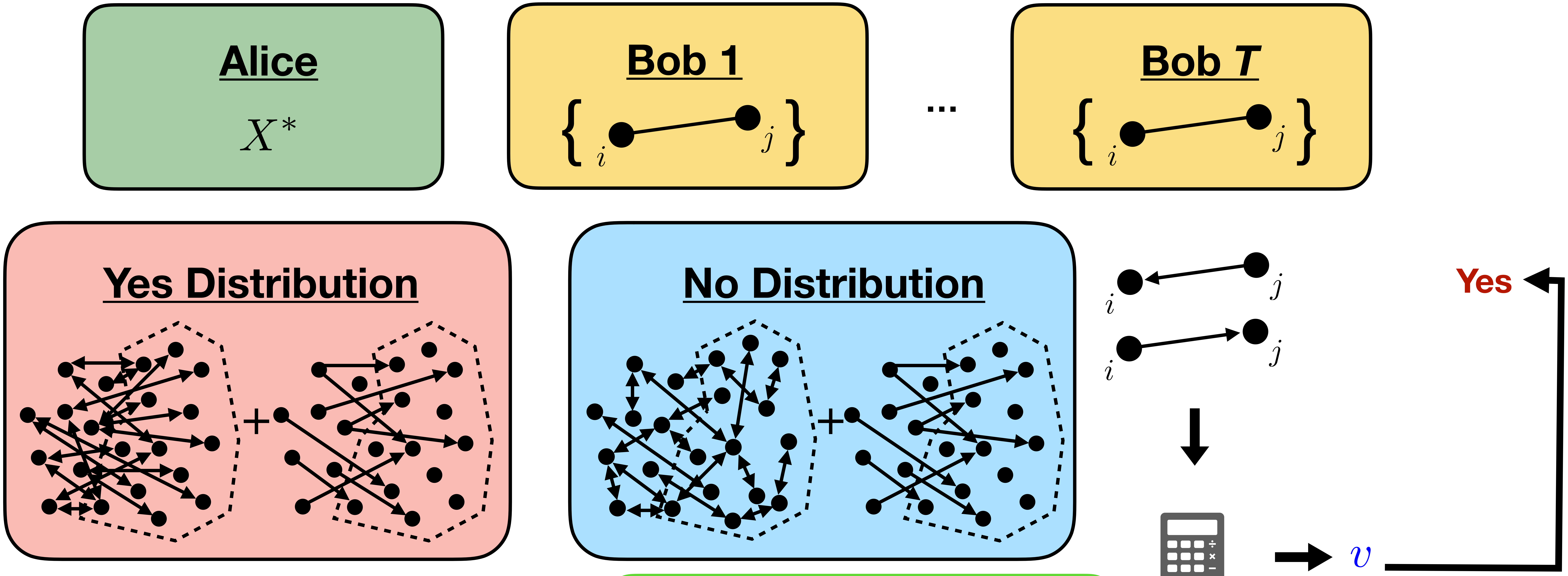

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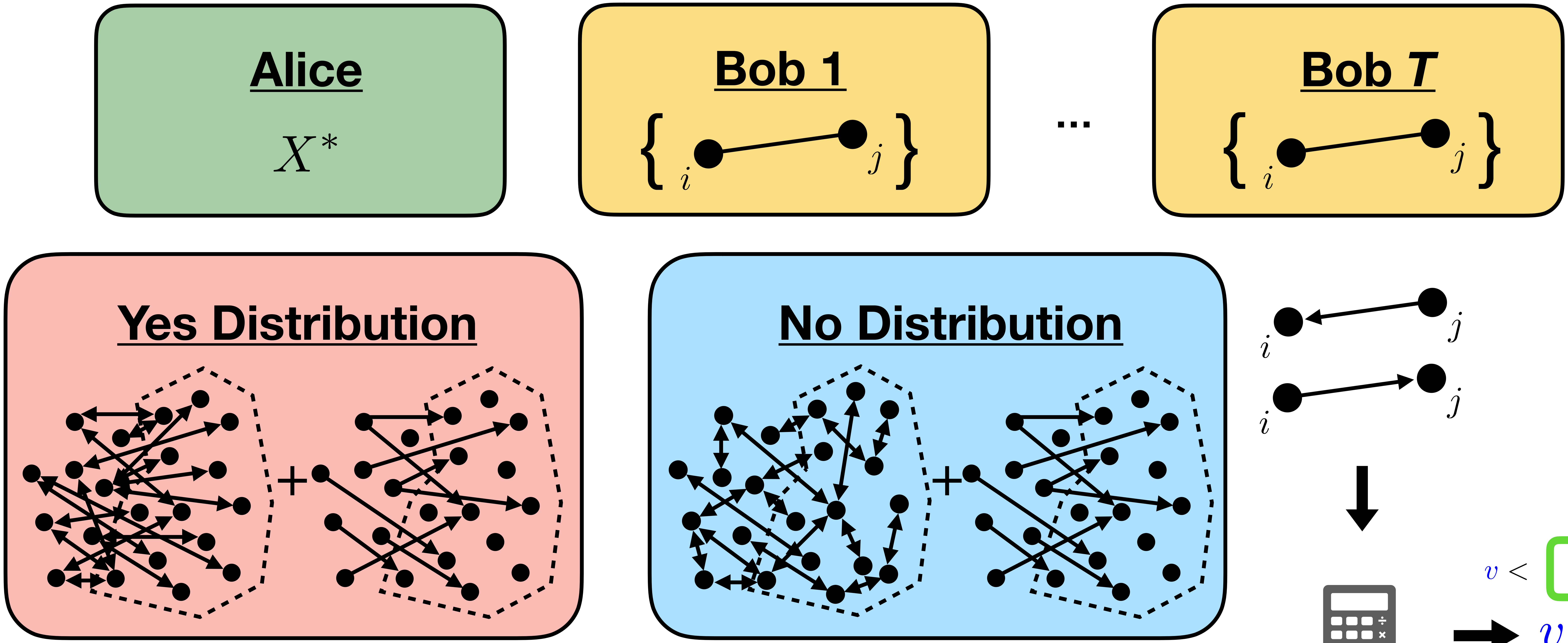

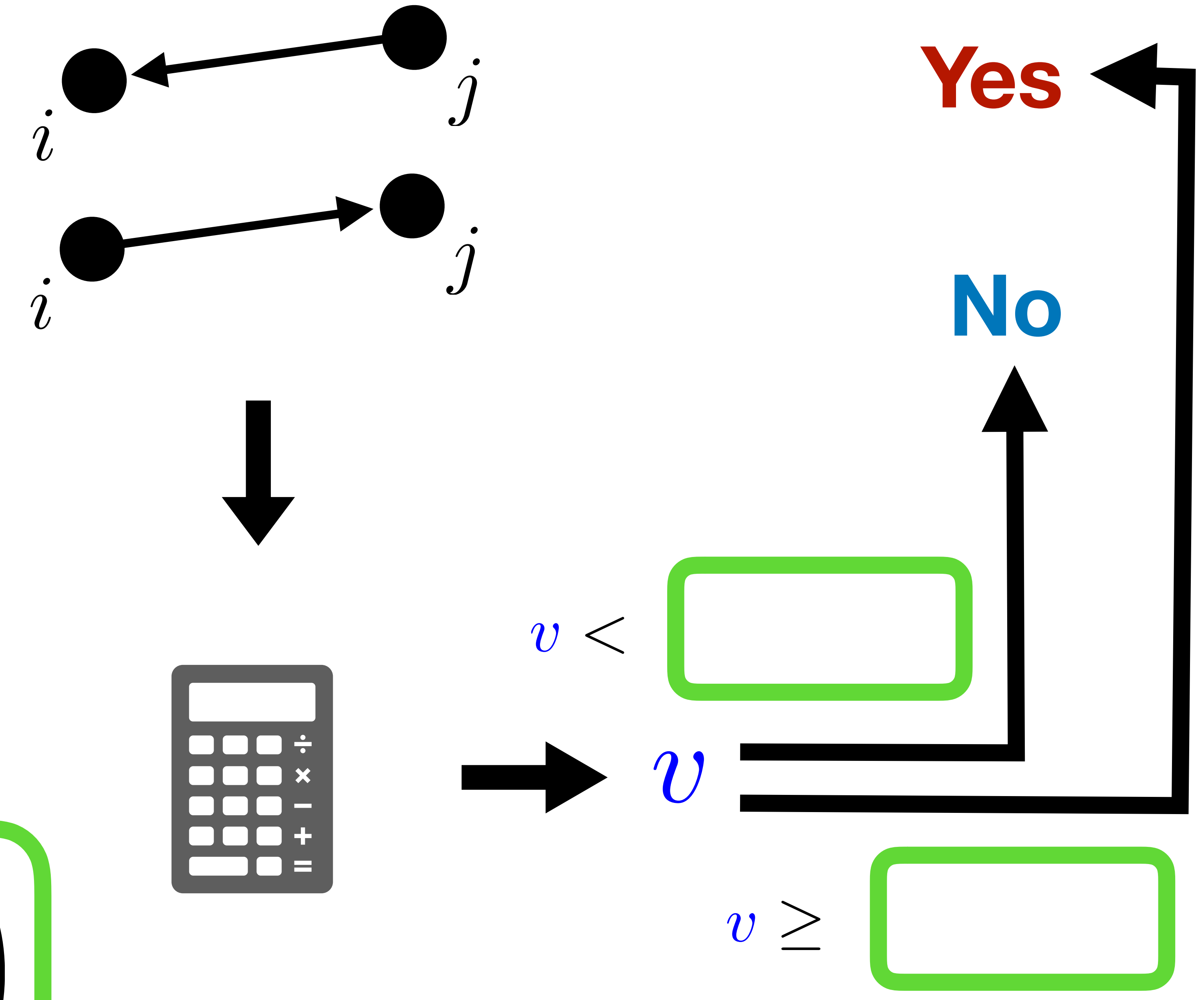

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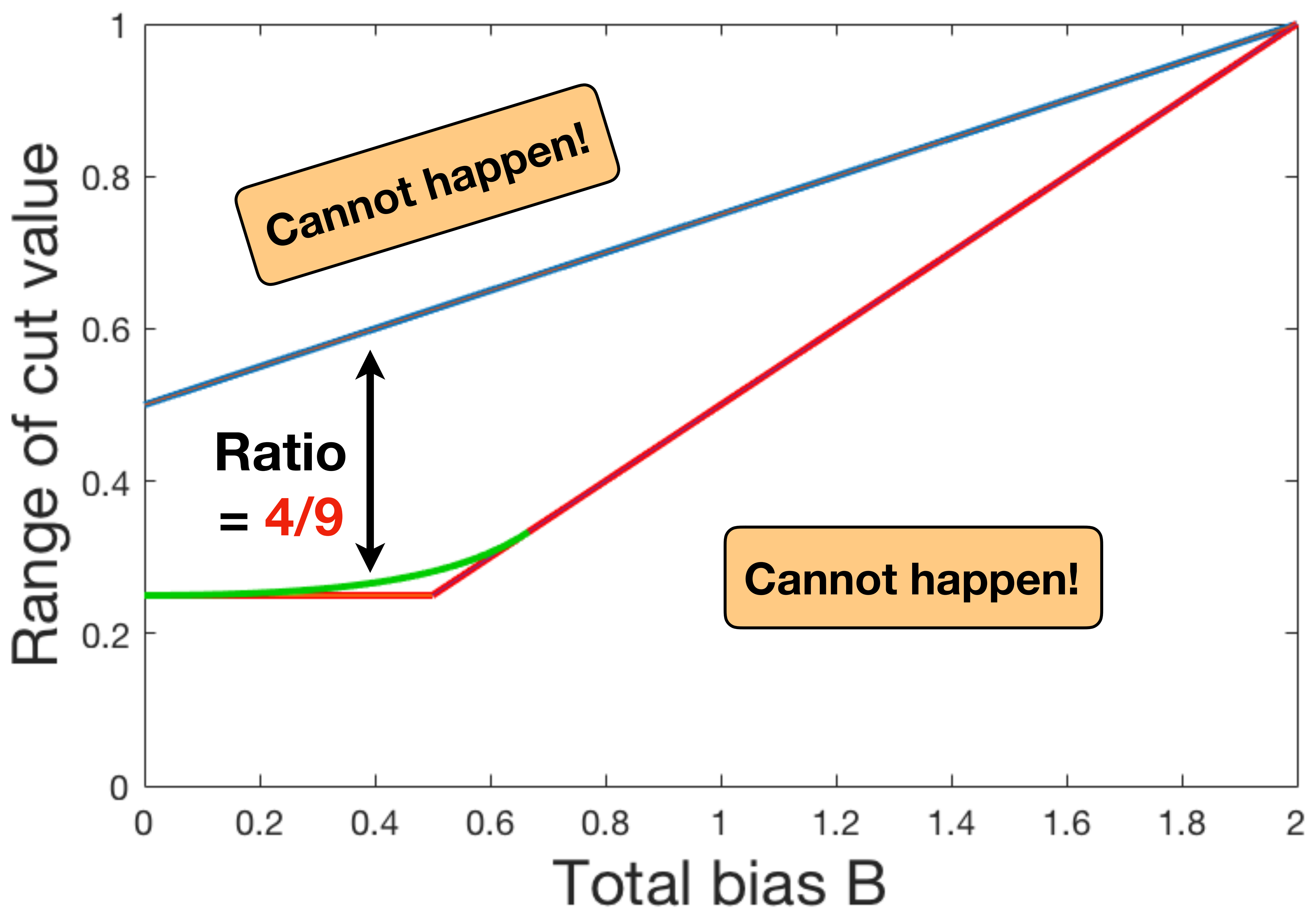

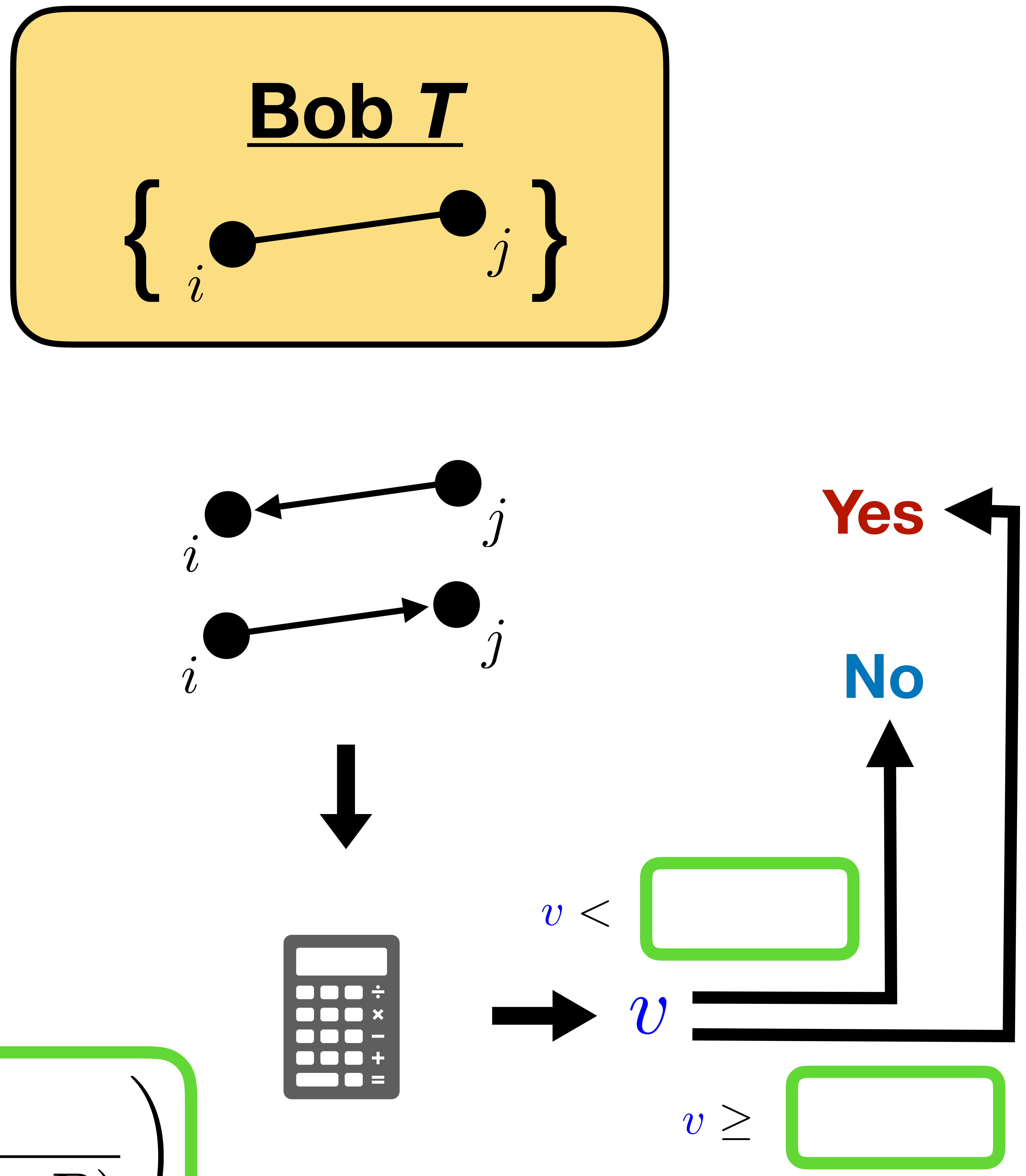

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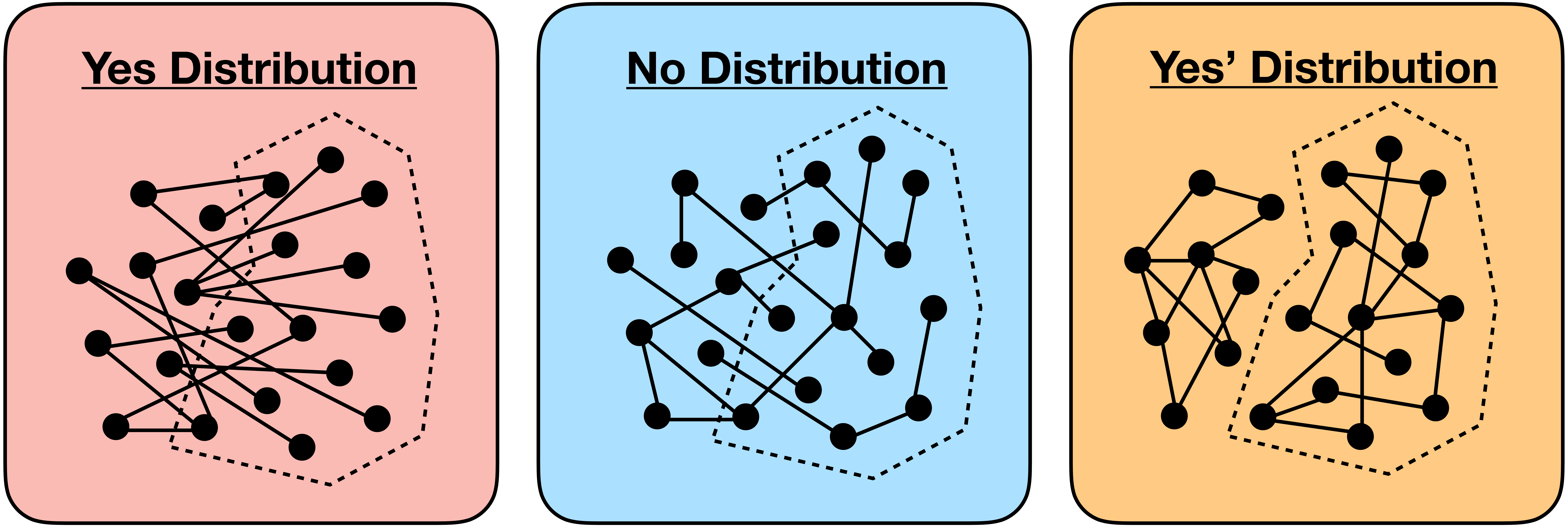

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Can be extended to Max k-SAT!

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Local random sampling is optimal!

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Thanks for your attention, questions?

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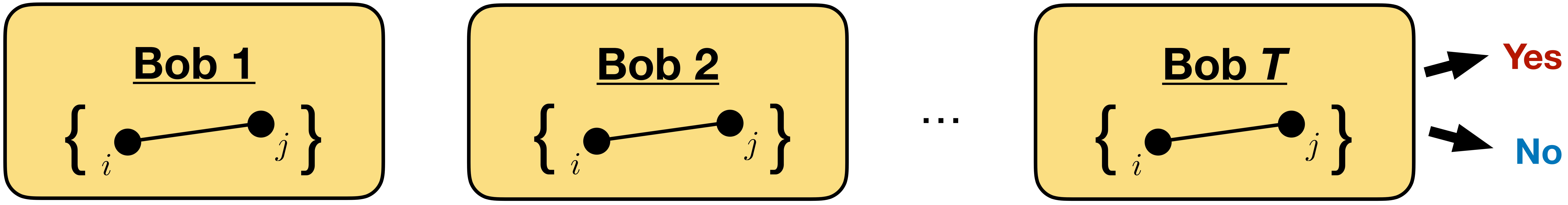

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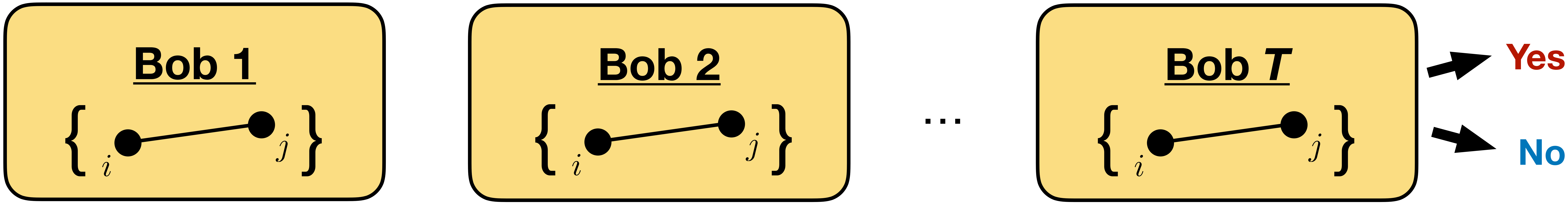

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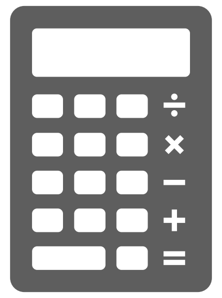

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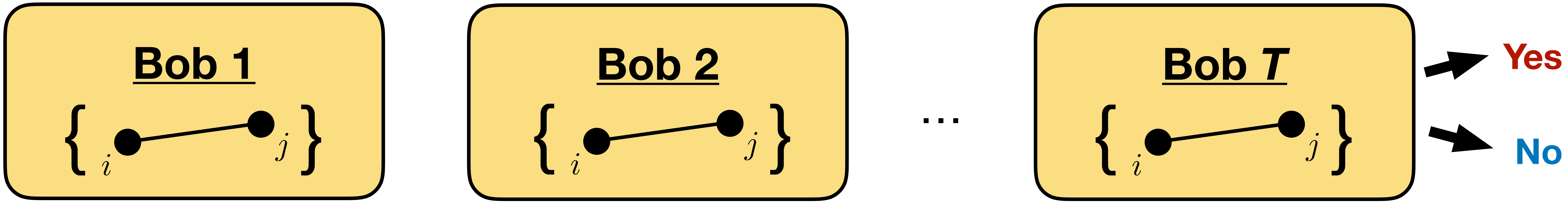

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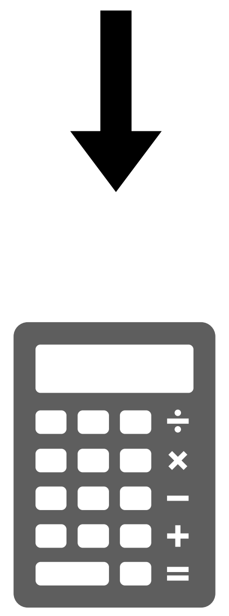

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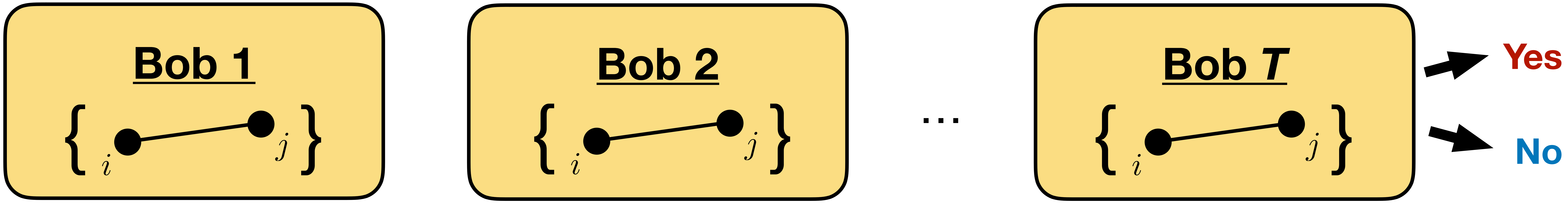

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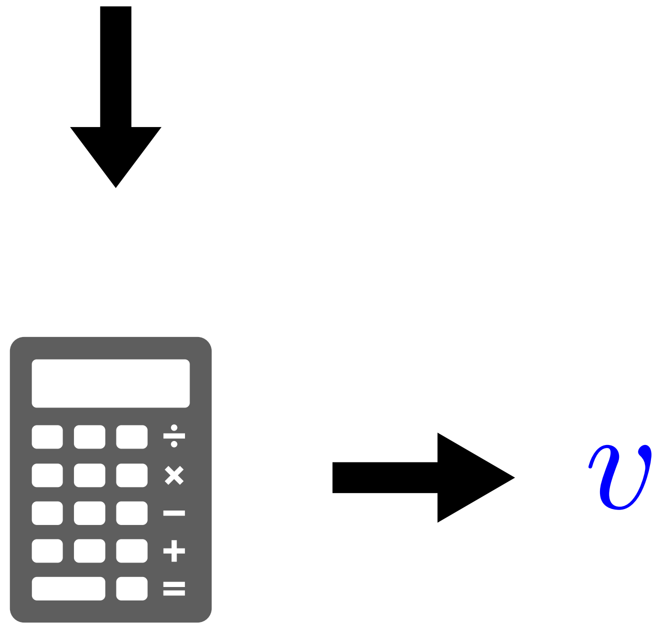

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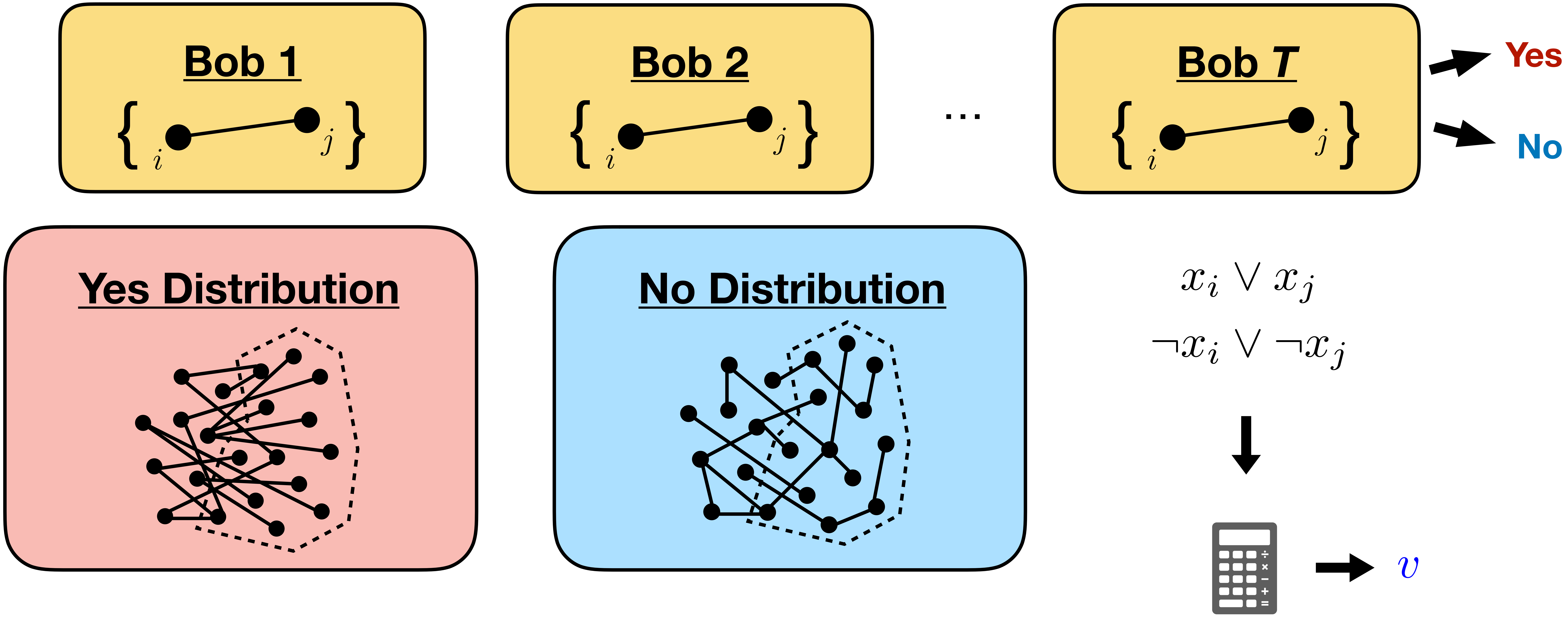

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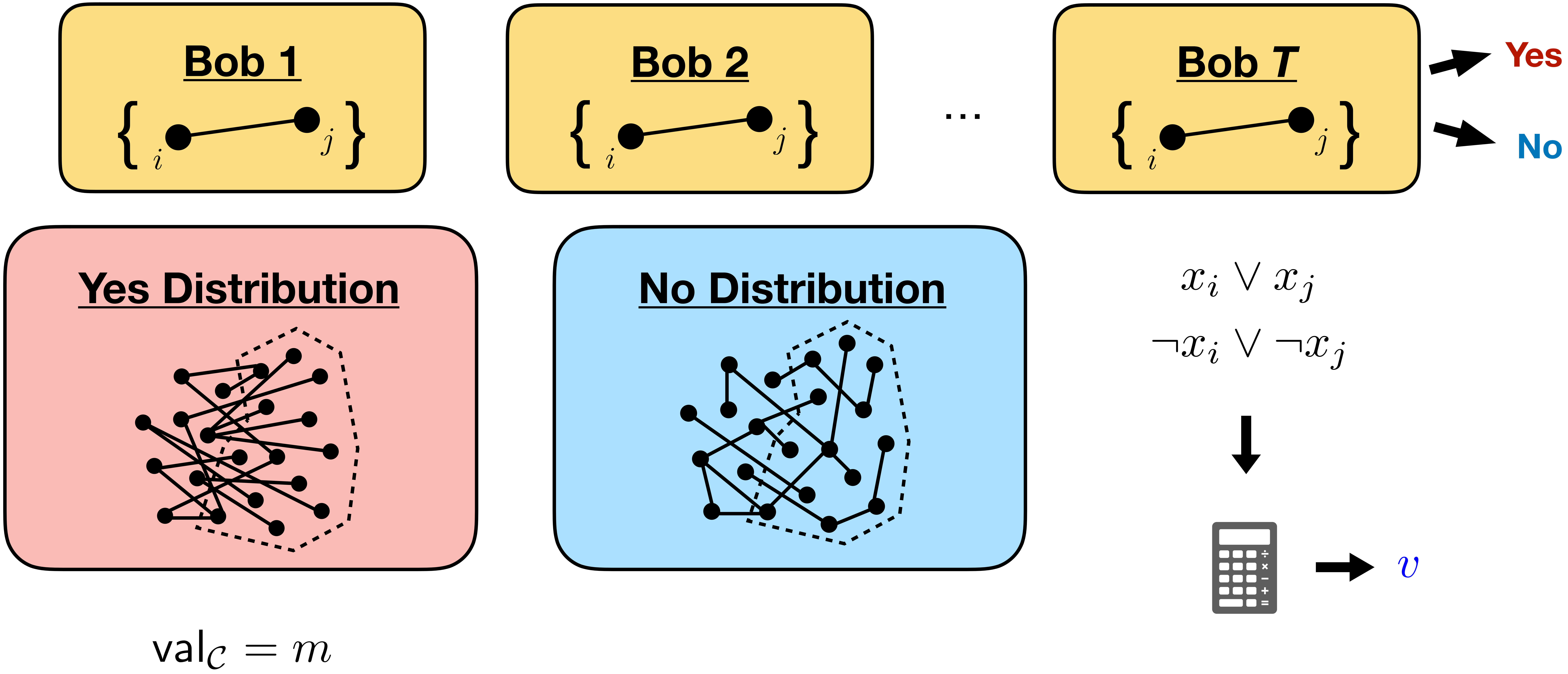
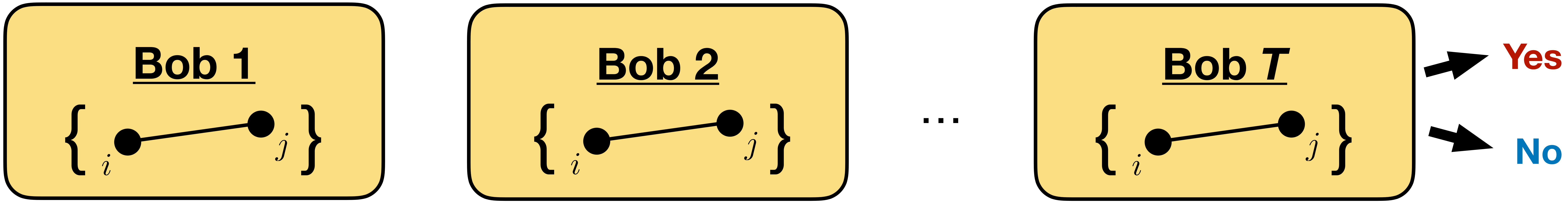
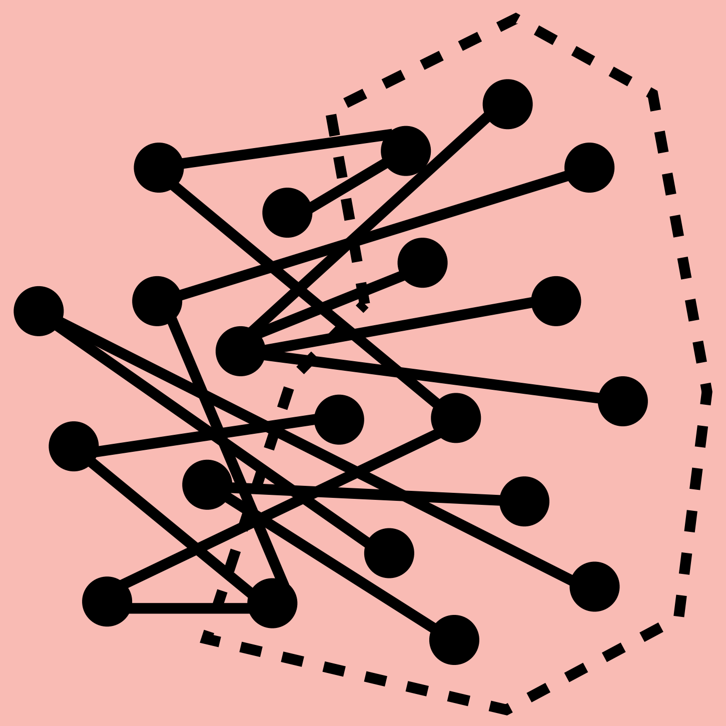


Reducing DBHP to Max-2EOR

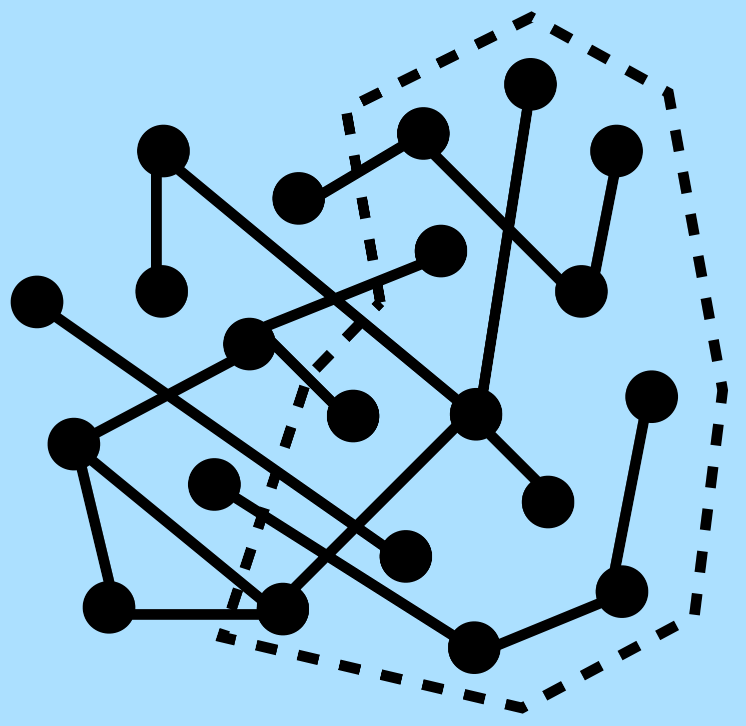


Yes Distribution



$\mathsf{val}_{\mathcal{C}} = m$

No Distribution



$\mathsf{val}_{\mathcal{C}} < \left(\frac{3}{4} + o(1)\right) \cdot m$

$x_i \vee x_j$

$\neg x_i \vee \neg x_j$

$\downarrow$

$\rightarrow v$

Reducing DBHP to Max-2EOR

Reducing DBHP to Max-2EOR

Max-2OR

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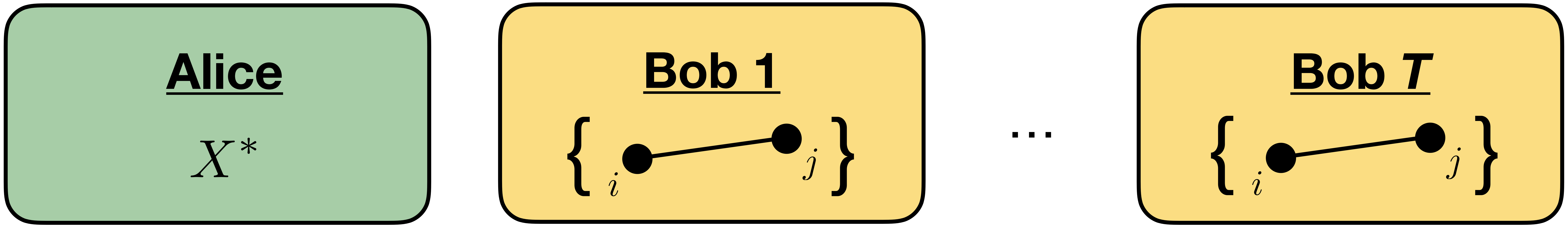
$$\mathsf{val}_{\mathcal{C}} = m$$

Reducing DBHP to Max-2OR

Reducing DBHP to Max-2OR

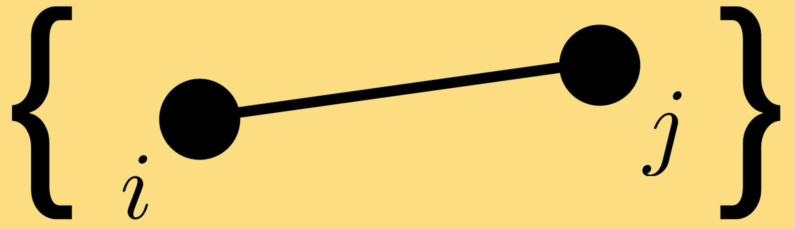


Reducing DBHP to Max-2OR

Alice

X^*

Bob 1



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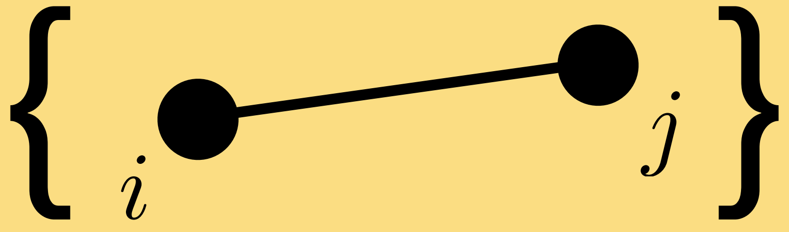
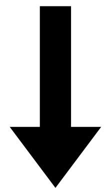
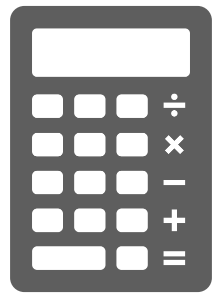
Bob *T*



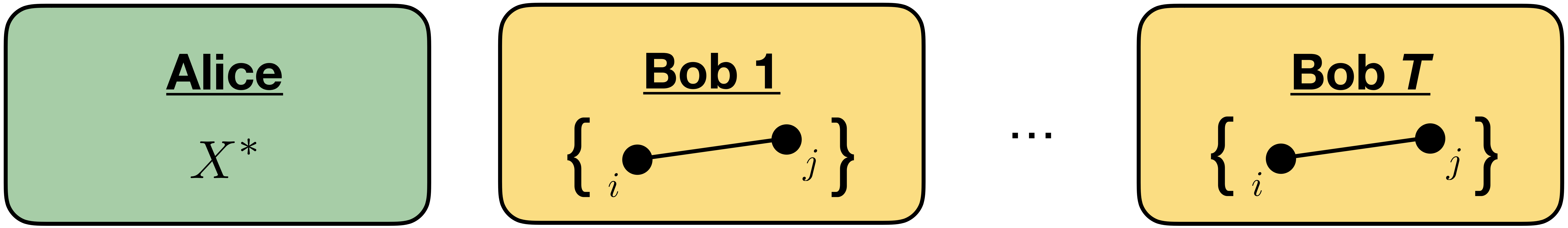
x_i

$\neg x_j$



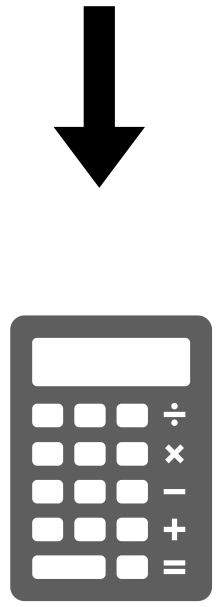


Reducing DBHP to Max-2OR

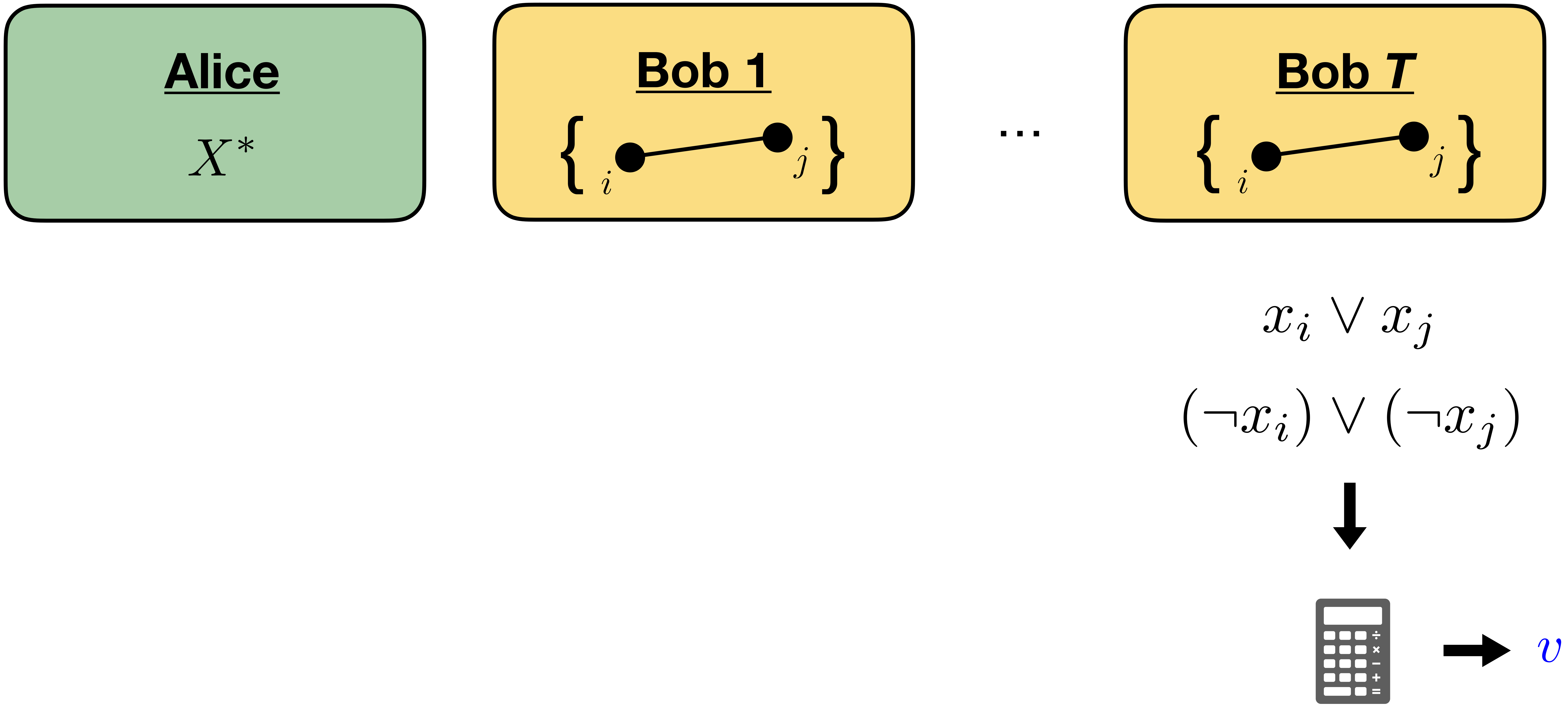


$$x_i \vee x_j$$

$$(\neg x_i) \vee (\neg x_j)$$



Reducing DBHP to Max-2OR



Reducing DBHP to Max-2OR

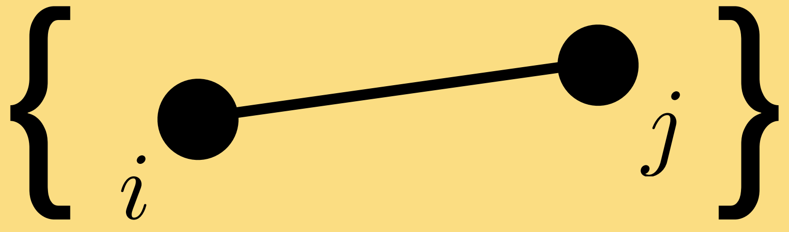
Alice

X^*

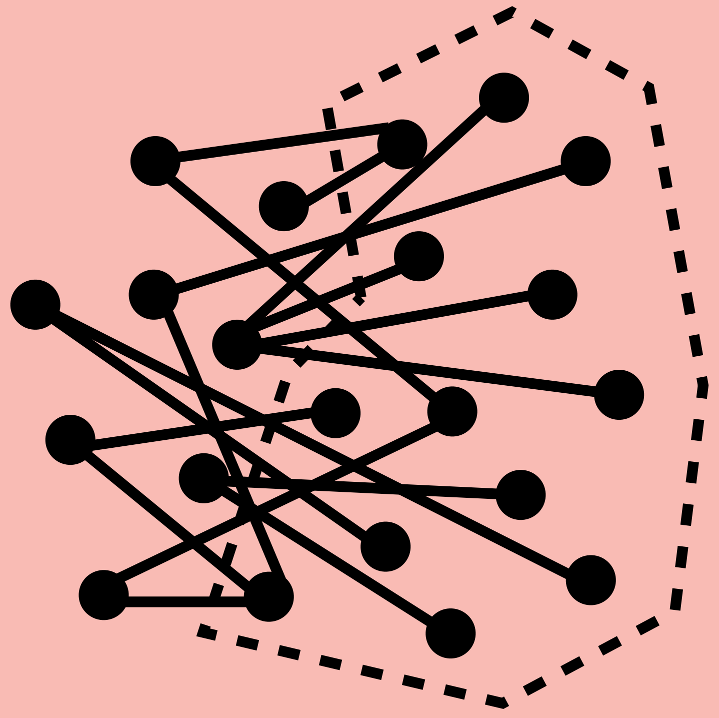
Bob 1

…

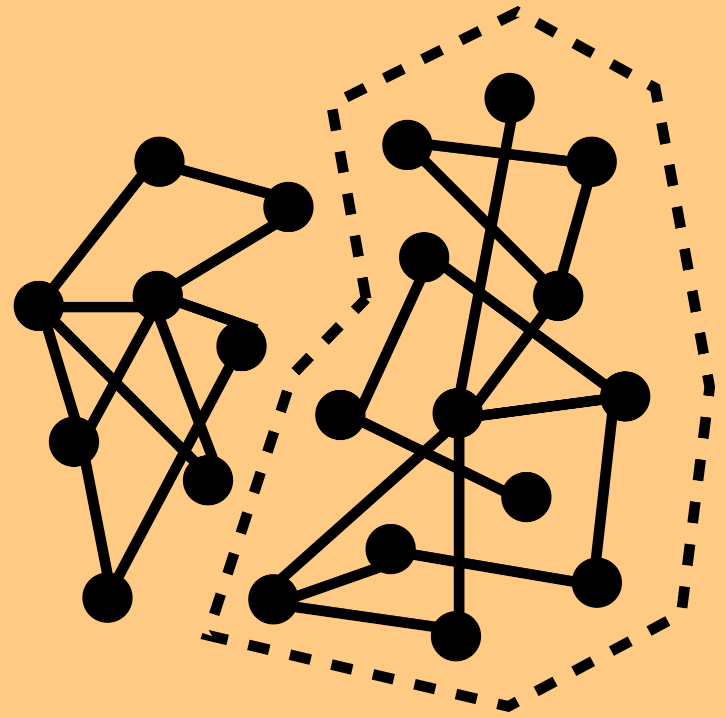
Bob T



Yes Distribution



Yes' Distribution



$x_i \vee x_j$

$(\neg x_i) \vee (\neg x_j)$

↓

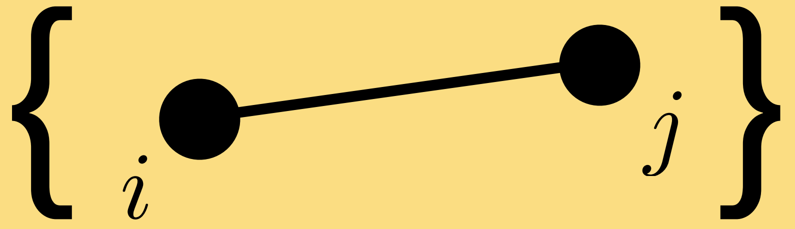
→ v

Reducing DBHP to Max-2OR

Alice

X^*

Bob 1



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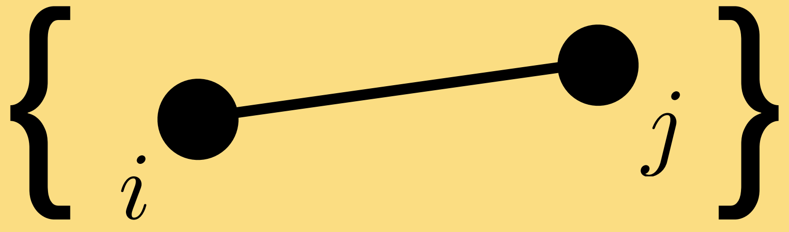
Bob *T*



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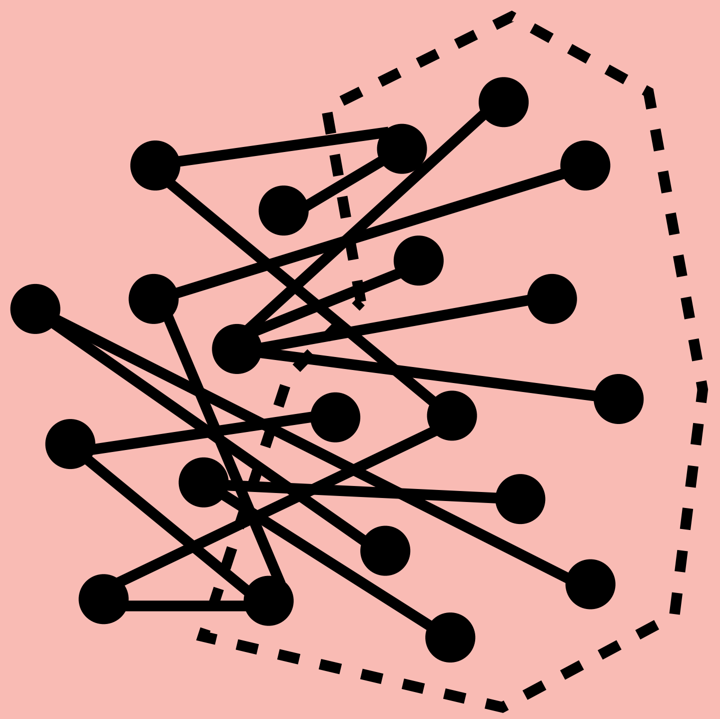
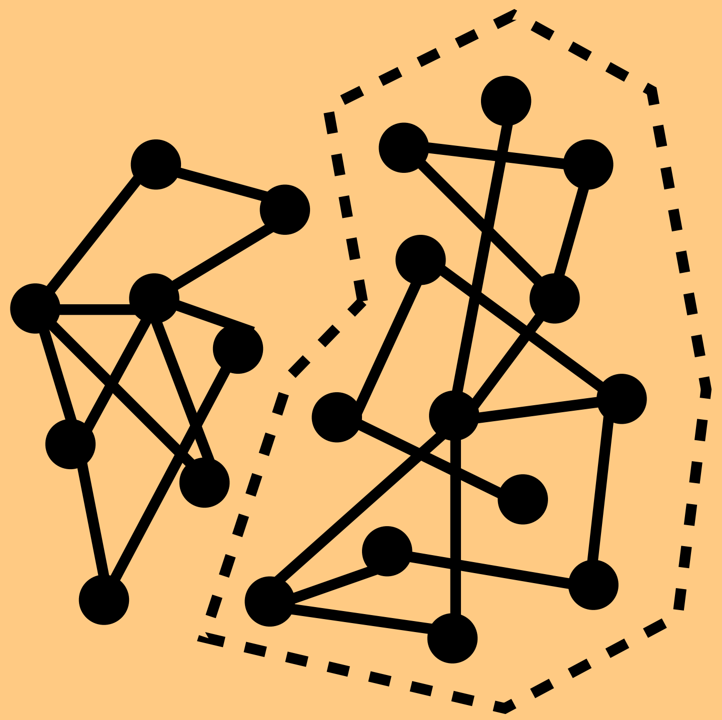


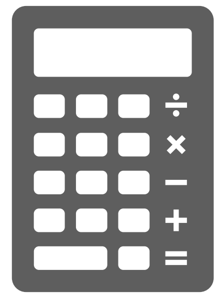
Yes' Distribution



$x_i \vee x_j$

$(\neg x_i) \vee (\neg x_j)$

↓



→ v

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Reducing DBHP to Max-2OR

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X^*

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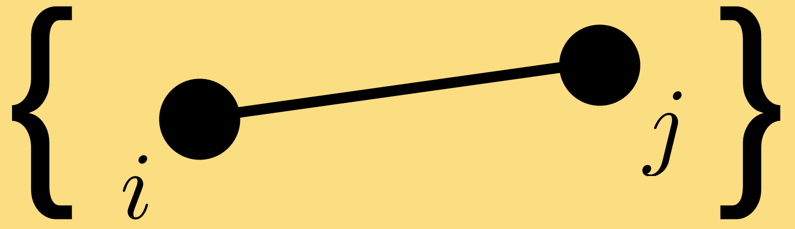

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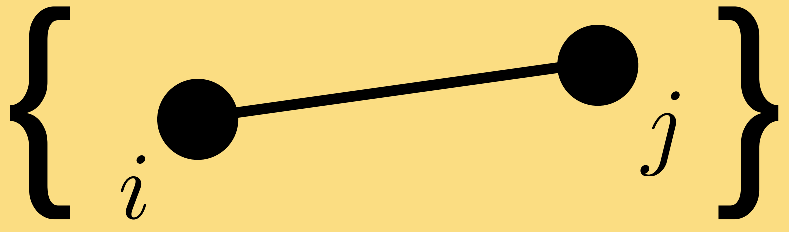

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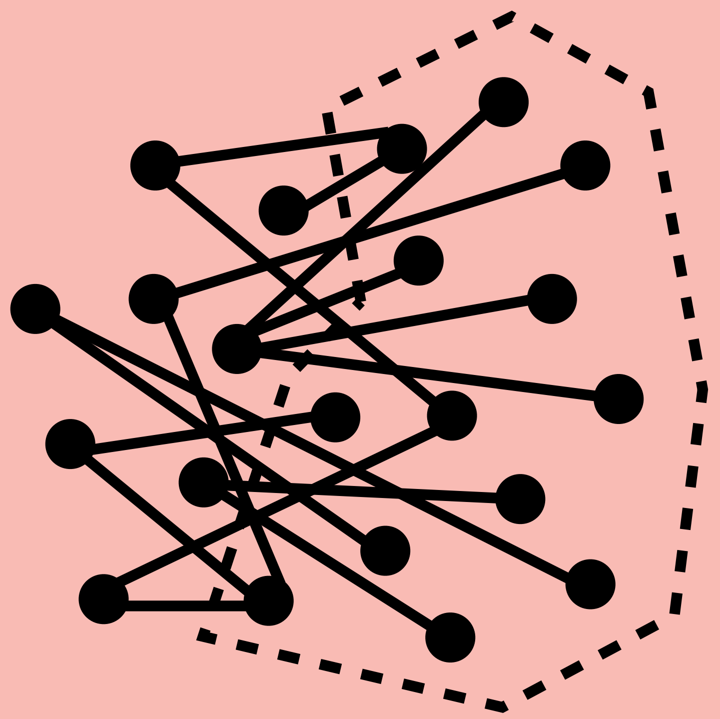

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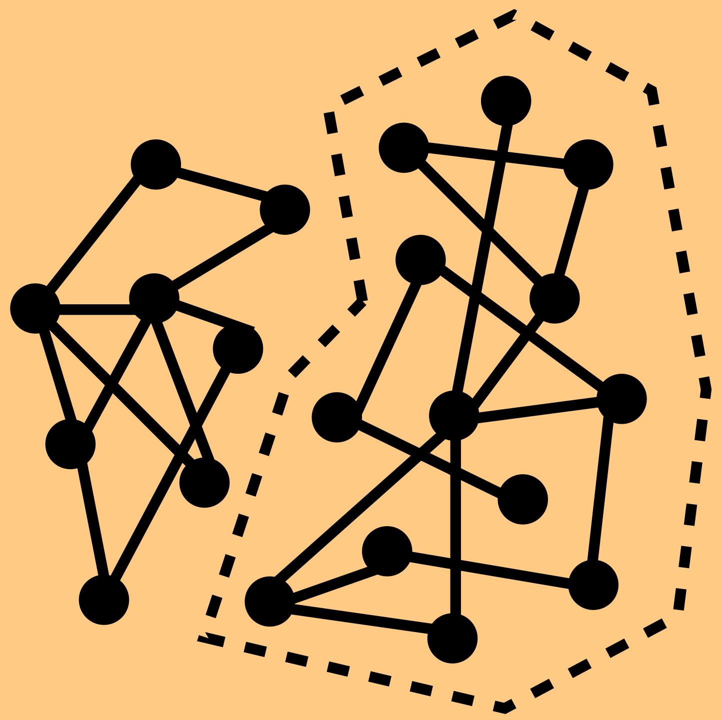
Yes' Distribution



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$x_i \vee x_j$

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v

$v < \left(\frac{\sqrt{2}}{2} + \epsilon\right) \cdot m$

Yes

Reducing DBHP to Max-2OR

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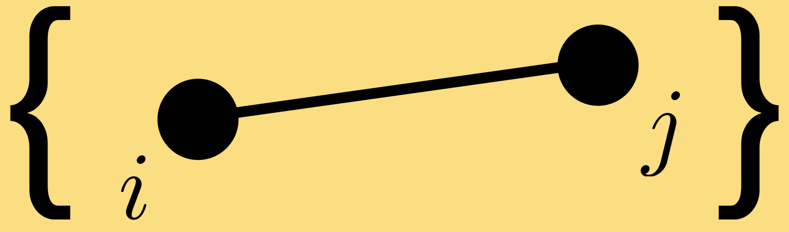

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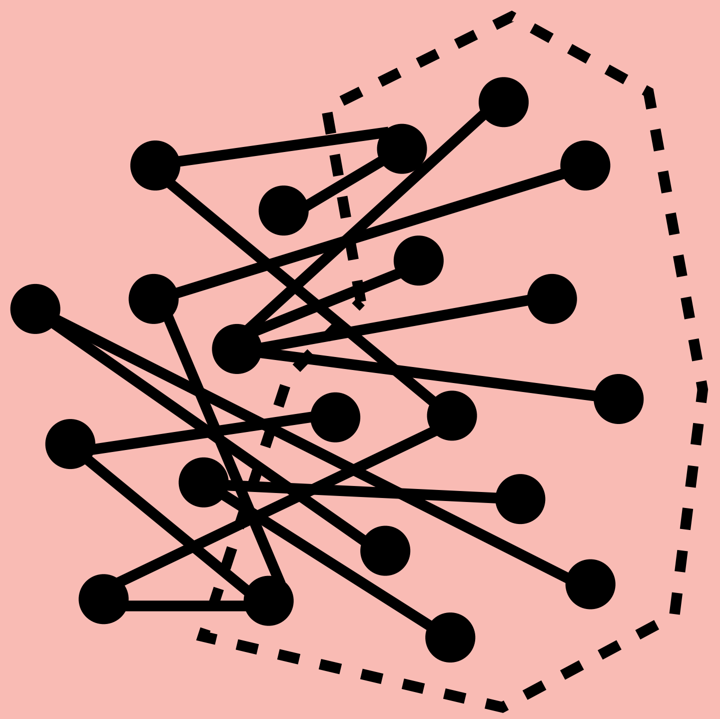

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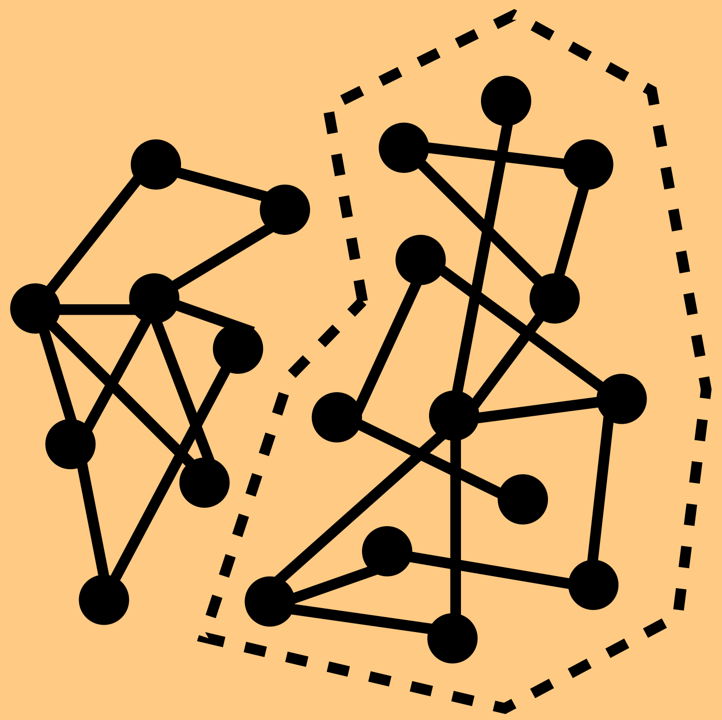

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$x_i \vee x_j$

$(\neg x_i) \vee (\neg x_j)$

v

$v < \left(\frac{\sqrt{2}}{2} + \epsilon\right) \cdot m$ → No

$v \geq \left(\frac{\sqrt{2}}{2} + \epsilon\right) \cdot m$ → Yes