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Subgraph Conditioning Method

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Subgraph conditioning method is a technique for characterizing a property in random graph model by conditioning on the number of small subgraphs. It mysteriously bypasses the obstacles encountered by naive analysis and results in many surprising results. In this notes, we will focus on a running example: random r-regular graph is asymptotic almost surely Hamiltonian for $r \geq 3$. We will start from the definition and properties of random regular graph model and only some background in provability is assumed.

1 Probability models for random regular graph

1.1 Pairing model

Pairing model introduced by Bollobás [Bol80] provides a convenient framework to analyze random regular graph. Let $r \geq 1$ and $n \in \mathbb{N}$ such that rn is even, generate a r-regular graph of n vertices as follows. First, start with rn vertices and group them into n cloud where each of them consists exactly r vertices. Next, pick a perfect matching among these rn vertices. We call such graph *cloud graph*. Finally, construct a graph of n vertices by adding edge between two vertices if and only if the corresponding clouds are connected by an edge. Note that the resulting graph might contain self-loop or multi-edge. We denote this distribution as $\mathcal{P}_{n,r}$.

Some observations immediately follow from the definition of pairing model.

- ($\mathcal{P}_{n,r}$ **is not uniform over all** r**-regular graph**) One can see that if a graph G contains self-loop or multi-edge, then probability of G being sampled in pairing model will increase by a factorial factor.
- ($\mathcal{P}_{n,r}$ **is uniform over all *simple*** r**-regular graph after conditioning**) We call a graph to be *simple* if it does not contain self-loop and multi-edge. One can see that when conditioning on G being simple, pairing model generates simple regular graph *uniformly*. We denote this conditional distribution as $\mathcal{R}_{n,r}$.

Ideally, we would hope for results in $\mathcal{R}_{n,r}$. However, it is much more complicated to analyze $\mathcal{R}_{n,r}$ comparing to $\mathcal{P}_{n,r}$. Thus, we will mainly focus on $\mathcal{P}_{n,r}$ in the following. To justify the sufficiency of analyzing $\mathcal{P}_{n,r}$, the following is a fact.

$$\mathbb{P}_{G\sim\mathcal{P}_{n,r}}[G \text{ is simple}] \sim e^{(1-r^2)/4}.$$

That is, for any constant r, simple graph has a constant probability to be sampled from $\mathcal{P}_{n,r}$. As a result, if one can show that an event E happens with probability $1 - o(1)$ in $\mathcal{P}_{n,r}$, it immediately implies that E happens with probability $1 - o(1)$ in $\mathcal{G}_{n,r}$.

1.2 Some notations

In this notes, we focus on the *asymptotic behavior* in the pairing model. That is, we consider the behavior when $n \to \infty$. For simplicity, we use $\xrightarrow{n\to\infty}$ to denote the asymptotic convergence of a sequence of random variables. We use *a.a.s.* as an abbreviation for *asymptotic almost surely*, which means that the probability of an event to happen goes to 1 when $n \to \infty$.

1.3 Small subgraphs in pairing model

After defining a probability model for regular graphs, one might be interested in showing whether random regular graph has some properties with high probability. For instance, whether a random regular graph is connected, has large girth, or is Hamiltonian. In this section, we are going to study the *number of small cycles* in random regular graphs. Surprisingly, we can characterize the asymptotic distribution of the number of small cycles as independent Poisson distribution. The following theorem was proved independently by Bollobás [Bol80] and Wormald [Wor80, Wor81].

Theorem 1 *Let $r \geq 1$ and $n \in \mathbb{N}$. For any $i \geq 1$, let $X_{i,n}$ denote the number of length i cycles in G where G is sampled from $\mathcal{P}_{n,r}$. For any constant $k \in \mathbb{N}$, we have*

$$(X_{1,n}, X_{2,n}, \dots, X_{k,n}) \xrightarrow{n\to\infty} (Z_1, Z_2, \dots, Z_k),$$

where $Z_i = Poi(\lambda_i)$ are independent Poisson distribution with mean $\lambda_i = \frac{(r-1)^i}{2i}$. Equivalently,

$$\lim_{n\to\infty} \mathbb{P}[\forall i \in [k],\ X_{i,n} = x_i] = \prod_{i\in[k]} \frac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!} \tag{1}$$

for any non-negative integers $x_1, x_2, \dots, x_k$.

Proof: The idea is based on the *moment method*. That is, showing that the joint moment of $X_{i,1}, X_{i,2}, \dots, X_{i,k}$ will converge to the joint moment of independent Poisson variables. Specifically, we are using the *Brun's sieve*, which is an important technique in Poisson approximation.

Lemma 2 (multivariate Brun's sieve) *Let $X_{1,n}, X_{2,n}, \dots, X_{k,n}$ be sequences of random variables. If for any non-negative integers $x_1, x_2, \dots, x_k$,*

$$\mathbb{E}[(X_{1,n})_{x_1} \cdot (X_{2,n})_{x_2} \cdots (X_{k,n})_{x_k}] \xrightarrow{n\to\infty} \prod_{i\in[k]} \lambda_i^{x_i}$$

for some non-negative $\lambda_1, \lambda_2, \dots, \lambda_k$. Then $X_{1,n}, X_{2,n}, \dots, X_{k,n}$ are asymptotic independent Poisson random variables with means λ_i.

Brun's sieve is somehow being treated as a folklore in the community. One can find a simplified proof in Chapter 8.3 of Alon and Spencer's book [AS04].

Now, let's proceed to show that the cycle counts satisfy the condition of Brun's sieve. For simplicity, we will only prove a basic case to get some taste. The complete proof can be easily generalized.

We are going to prove the following.

$$\mathbb{E}[X_{i,n}] \xrightarrow{n\to\infty} \lambda_i.$$

The idea is to rewrite the expectation as

$$\mathbb{E}[X_{i,n}] = \sum_{C:\ C \text{ is a length } i \text{ cycle in the cloud graph}} \mathbb{P}[C]$$
$$= |\{C:\ C \text{ is a length } i \text{ cycle in the cloud graph}\}| \cdot \mathbb{P}[\mathbb{C}].$$

To estimate the first term, we first count the number of choices of length i ordered cycles in the cloud graph.

$$n \cdot (n-1) \cdot (n-i+1) \cdot [r(r-1)]^i \xrightarrow{n\to\infty} [r(r-1)n]^i.$$

By symmetry and rotation, we overcount for $2i$ times so the correct number of distinct choices is

$$|\{C:\ C \text{ is a length } i \text{ cycle in the cloud graph}\}| \xrightarrow{n\to\infty} \frac{[r(r-1)n]^i}{2i}.$$

Now, consider the probability of certain set of $2i$ vertices in the cloud graph being selected.

$$\mathbb{P}[C] = \frac{(rn-2i-1)\cdot(rn-2i-3)\cdots 1}{(rn-1)\cdot(rn-3)\cdots 1} \xrightarrow{n\to\infty} (rn)^i,$$

where the denominator is the number of cloud graphs and the numerator is the number of cloud graphs that contain C. By multiplying the two, we have $\mathbb{E}[X_{i,n}] \xrightarrow{n\to\infty} \frac{(r-1)^i}{2i} = \lambda_i$ as desired.

To generalize the proof, one need to consider more complicated cases where different cycles might share a same vertex. However, it can be shown that the number of cloud graphs having such properties is much less than the case where every cycle have no common vertex. □

Another interesting fact about subgraph in random regular graph is that a random regular graph does not contain a dense subgraph asymptotically.

Theorem 3 *Let $r \geq 1$ and $n \in \mathbb{N}$. Suppose H is a constant-size subgraph with the number of edges more than that of vertices. Then G does not contain H a.a.s. when $n \to \infty$ where G is sampled from $\mathcal{G}_{n,r}$.*

PROOF: Let H be a subgraph of v vertices and e edges where v, e are constant with respect to n. Let X_n be the number of H appears in G sampled from $\mathcal{P}_{r,n}$. With the similar argument as in the proof of Theorem 1, we have

$$\mathbb{E}X_n \lesssim n\cdot(n-1)\cdot(n-v+1)\cdot(\frac{1}{rn})^e \lesssim n^{v-e} \xrightarrow{n\to\infty} 0.$$

Also,

$$\mathbb{E}X_n^2 \lesssim n\cdot(n-1)\cdot(n-2v+1)\cdot(\frac{1}{rn})^{2e} \lesssim n^{2v-2e} \xrightarrow{n\to\infty} 0.$$

By Chebyshev's inequality, we conclude that H does not appear in G a.a.s. □

2 Subgraph conditioning method

2.1 High-level idea

Here we introduce the high-level idea of subgraph conditioning method and use the number of Hamiltonian cycles in random regular graph as a motivating example. Let Y_n be the random variable of interest (here $Y_n = H_n$, the number of Hamiltonian cycles). The goal is to show that $Y_n > 0$ a.a.s.. However, it happens that the naive way of applying Chebyshev inequality does not work. For instance, the asymptotic mean and the second moment of H_n are the following (we omit the proof for simplicity).

$$\mathbb{E}[H_n] \sim e\sqrt{\frac{\pi}{2n}}\Big(\frac{(r-2)^{r-2}(r-1)^2}{r^{r-2}}\Big)^{n/2},\ \frac{\mathbb{E}H_n^2}{(\mathbb{E}H_n)^2} \xrightarrow{n\to\infty} \frac{r}{(r-2)e^{2/(r-1)}}.$$

That is, for any constant r, one can only get $\mathbb{P}[H_n > 0]$ with some small probability which does not go to 1 with $n \to \infty$.

Now, the subgraph conditioning method comes to our rescue. The idea here is conditioning on some other random variables $X_{1,n}, X_{2,n}, \dots, X_{k,n}$ defined on the graph (here $X_{i,n}$ be the number of cycles of length i). Namely, writing $Y_n = Y_n' + (Y_n - Y_n')$ where Y_n' is $Y_n' = Y_n | X_{1,n}, \dots, X_{k,n}$. Mysteriously, usually we will have the following two properties.

- $(Y_n - Y_n')$ is very small (does not grow with n) a.a.s..
- Y_n' converges to some random variable in $X_{1,n}, \dots, X_{k,n}$ and is greater than 0 a.a.s..

With these two properties, we then can conclude that $Y_n > 0$ a.a.s..

So far we have been pretty vague and thus we are going to make the above ideas clear in the following subsections.

2.2 Asymptotic conditional distribution

As we discussed in the previous subsection, the subgraph conditioning method tells us that one can write Y_n into random variables that depends on $X_{i,n}$ asymptotically. The following theorem provides a sufficient condition to achieve this goal.

Theorem 4 *Let $\lambda_i > 0$ and $\delta_i \geq -1$ for $i = 1, 2, \dots,$ be constants. Suppose there are non-negative integer random variables $X_{1,n}, X_{2,n}, \dots$ and Y_n such that*

1. *$X_{i,n} \xrightarrow{n\to\infty} Z_i$ where $Z_i = Poi(\lambda_i)$;*
2. *For any non-negative integers $x_1, x_2, \dots, x_k$,*

$$\frac{\mathbb{E}[Y_n \mid X_{1,n} = x_1, X_{2,n} = x_2, \dots, X_{k,n} = x_k]}{\mathbb{E}Y_n} \xrightarrow{n\to\infty} \prod_{i\in[k]} (1+\delta_i)^{x_i} e^{-\lambda_i(1+\delta_i)} e^{\lambda_i}, \tag{2}$$

or equivalently,

$$\frac{\mathbb{E}[Y_n \cdot (X_{1,n})_{x_1} \cdot (X_{2,n})_{x_2} \cdots (X_{k,n})_{x_k}]}{\mathbb{E}Y_n} \xrightarrow{n\to\infty} \prod_{i\in[k]} [\lambda_i(1+\delta_i)]^{x_i}; \tag{3}$$

3. *$\sum_{i=1}^{\infty} \lambda_i \delta_i^2 < \infty$;*
4. *$\frac{\mathbb{E}Y_n^2}{(\mathbb{E}Y_n)^2} \leq \exp(\sum_{i=1}^{\infty} \lambda_i \delta_i^2)$ when $n \to \infty$.*

Then

$$\frac{Y_n}{\mathbb{E}Y_n} \xrightarrow{n\to\infty} W \sim \prod_{i=1}^{\infty}(1+\delta_i)^{Z_i}e^{-\lambda_i(1+\delta_i)}e^{\lambda_i}. \tag{4}$$

The proof of Theorem 4 will be postponed to Section 2.5. Here, we are going to see some remarks about the theorem and then see an example in the next subsection.

- Let's start with the most confusing term in the theorem: δ_i. Intuitively, one can simply ignore the annoying 1 there and think of $\mu_i = (1+\delta_i) \geq 0$ as the frequency parameter of a Poisson random variable. From this view, (2) and (4) can be interpreted as saying $\frac{\mathbb{E}[Y_n \mid X_{i,n}=x_i,\ \forall i\in[k]]}{\mathbb{E}Y_n}$ (resp. $\frac{Y_n}{\mathbb{E}Y_n}$) be proportional to $\prod_{i\in[k]}\mu_i^{x_i}$ (resp. to $\prod_{i\in[k]}\mu_i^{Z_i}$) asymptotically.
- The case where $\mu_i = 0$, *i.e.,* $\delta_i = -1$, is also confusing at first glance. A way to interpret it is as follows. When $Z_i > 0$, then $W = 0$. Note that this does not mean $Y_n = 0$ when $X_{i,n} = 0$.
- In the case where $\mu_i > 0$, *i.e.,* $\delta > -1$, one can see that the corresponding term is always positive.

From the above discussion, one can see that when $Z_i = 0$ for all i such that $\mu_i = 0$, *i.e.,*$\delta_i = -1$, $W > 0$ and thus $Y_n > 0$. Namely, $\lim_{n\to\infty}\mathbb{P}[Y_n > 0] \geq \mathbb{P}[Z_i = 0,\ \forall i,\ \delta_i = -1]$.

2.3 An example: Random regular graph is Hamiltonian

In this subsection, we are going to use Theorem 4 to show that random r-regular graph is Hamiltonian a.a.s. when $r \geq 3$.

Let's first recall why the naive way failed in proving this. With some effort, one can show that the first and second moments of H_n are the following.

$$\mathbb{E}[H_n] \sim e\sqrt{\frac{\pi}{2n}}\Big(\frac{(r-2)^{r-2}(r-1)^2}{r^{r-2}}\Big)^{n/2},\ \frac{\mathbb{E}H_n^2}{(\mathbb{E}H_n)^2} \xrightarrow{n\to\infty} \frac{r}{(r-2)e^{2/(r-1)}}.$$

By simply applying Chebyshev's inequality, one can only get constant probability, which is not the asymptotic almost surely guarantee. The main barrier lies in the second moment: the variance is too large.

Nevertheless, the subgraph conditioning method comes to rescue and we have the following theorem by Janson [Jan95].

Theorem 5 *Let $r \geq 3$ be a constant and let H_n be the number of Hamiltonian cycles in a graph G sampled from $\mathcal{P}_{n,r}$. Then*

$$\frac{H_n}{\mathbb{E}H_n} \xrightarrow{n\to\infty} \prod_{i\geq 3,\ i\ odd}\Big(1-\frac{2}{(r-1)^i}\Big)^{Z_i}e^{1/i},$$

where $Z_i \sim Poi(\frac{(r-1)^i}{2i})$ are independent Poisson random variables.

From Theorem 4, we know that to prove Theorem 5, it suffices to show the following.

$$\frac{\mathbb{E}H_n(X_{i,n})_{x_i}}{\mathbb{E}H_n} \xrightarrow{n\to\infty} \mu_i^{x_i}$$

for any odd $i \geq 3$ and $x_i \geq 0$.

Proof of Theorem 5: Here we only prove the special case where $x_i = 1$. The general cases follow similar idea with some elaboration. Specifically, we are going to prove

$$\frac{\mathbb{E}H_n X_{i,n}}{\mathbb{E}H_n} \xrightarrow{n\to\infty} \mu_i$$

for odd $i \geq 3$. The proof is still quite complicated and thus here we will only see the sketch.

The first step is based on an observation of symmetry. Let $\tilde{G}$ be a cloud graph and G be its corresponding regular graph. Define $\tilde{C}_i$ be the set of cloud graphs that correspond to a length i cycle. Note that $\tilde{C}_n$ contains the cloud graphs that correspond to to a Hamiltonian cycle. The following probability is over $\mathcal{P}_{n,r}$.

$$\begin{aligned}\frac{\mathbb{E}H_n X_{i,n}}{\mathbb{E}H_n} &= \frac{\sum_{\tilde{C}\in\tilde{C}_i}\sum_{\tilde{H}\in\tilde{C}_n}\mathbb{P}[\tilde{C}\subseteq\tilde{G}]}{\sum_{\tilde{H}\in\tilde{C}_n}\mathbb{P}[\tilde{H}\subseteq\tilde{G}]}\\ &= \frac{1}{|\tilde{C}_n|}\sum_{\tilde{C}\in\tilde{C}_i}\mathbb{P}[\tilde{C}\subseteq\tilde{G}\mid\tilde{H}\subseteq\tilde{G}],\end{aligned}$$

where $\tilde{H}$ is an arbitrary element in $\tilde{C}_n$ and the second equality is due to the symmetry in $\tilde{C}_n$.

Now, the job left is to estimate $|\tilde{C}_n|$ and $\mathbb{P}[\tilde{C}\subseteq\tilde{G}\mid\tilde{H}\subseteq\tilde{G}]$, which is quiet cumbersome. For those who are interested, please refer to [Jan95]. □

2.4 Contiguity

Contiguity is one of the central idea in subgraph conditioning method. In this subsection, we will first define what contiguity is and then give several examples as well as some connection to the previous discussion.

Definition 6 (contiguity) *Let $\mathcal{G}_n$ and $\mathcal{G}'_n$ be two random graph models, we say they are contiguous if for any sequence of event A_n, A_n is a.a.s. in $\mathcal{G}_n$ if and only if A_n is a.a.s. in $\mathcal{G}'_n$. We denote them as $\mathcal{G}_n \approx \mathcal{G}'_n$.*

Contiguity is a very useful notion to illustrate results in subgraph conditioning method. Let's translate Theorem 4 into the language of contiguity. Let Y_n be a sequence of random variables and $\mathcal{G}_n$ be a random graph model. Define $\mathcal{G}_n^{(Y_n)}$ be a new random graph model with probability distribution $\mathbb{P}_{\mathcal{G}_n}[G]\cdot\frac{Y_n(G)}{\mathbb{E}Y_n}$ for any graph G of n vertices. The following is an immediately corollary of Theorem 4.

Corollary 7 *Suppose the conditions in Theorem 4 hold, then $\overline{\mathcal{P}_{n,r}} \approx \overline{\mathcal{P}_{n,r}}^{(Y_n)}$, where $\overline{\mathcal{G}}$ denote the random graph model conditioning on the event $\{X_{i,n}=0 \mid \forall i,\ \delta_i=-1\}$.*

In addition to the above corollary, there are also some interesting facts listed below without proof. In the following, $\mathcal{H}_n$ denotes the random graph model that generates Hamiltonian cycles uniformly. Also, $\mathcal{G}_N \oplus \mathcal{G}'_n$ is the *graph-restricted sum* defined as summing the edges sampled from the two models and reject when encountering multi-edge.

Lemma 8 (fun facts about contiguity)

- *For any constant* $r \geq 3$*,* $\mathcal{G}_{n,d-1} \oplus \mathcal{G}_{n,1} \approx \mathcal{G}_{n,d}$.
- *For any constant* $r \geq 3$*,* $\mathcal{G}_{n,d-2} \oplus \mathcal{G}_{n,2} \approx \mathcal{G}_{n,d}$.
- $\mathcal{G}_{n,1} \oplus \mathcal{G}_{n,1} \not\approx \mathcal{G}_{n,2}$.
- *For any constant* $r \geq 3$*,* $\mathcal{G}_{n,d-2} \oplus \mathcal{H}_n \approx \mathcal{G}_{n,d}$.
- $\mathcal{H}_n \oplus \mathcal{H}_n \approx \mathcal{G}_{n,4}$.

2.5 Proof of Theorem 4

Proof of Theorem 4: Here we omit the analytic details and focus on the structure of the proof. The idea is simply considering the following conditional random variable of Y_n on $X_{i,n}$. For any $m \in \mathbb{N}$, define

$$Y_n^{(m)} = \mathbb{E}[Y_n \mid X_{1,n}, X_{2,n}, \dots, X_{m,n}].$$

Note that, $Y_n^{(m)}$ is a random variable in $X_{1,n}, X_{2,n}, \dots, X_{m,n}$.

Recall that our goal is to show that $Y_n \xrightarrow{n\to\infty} W$ where $W \sim \prod_{i=1}^{\infty}(1+\delta_i)^{Z_i} e^{-\lambda_i(1+\delta_i)} e^{\lambda_i}$. Similarly, for any $m \in \mathbb{N}$, define

$$W^{(m)} \sim \prod_{i=1}^{m}(1+\delta_i)^{Z_i} e^{-\lambda_i(1+\delta_i)} e^{\lambda_i}$$

Now, we can show $Y_n \xrightarrow{n\to\infty} W$ be triangle inequality. For any $\epsilon > 0$, we have

$$\limsup_{n\to\infty} \mathbb{P}\Big[|Y_n - W| > 3\epsilon\Big] \leq \limsup_{n\to\infty} \mathbb{P}\Big[|Y_n - Y_n^{(m)}| > \epsilon\Big] + \limsup_{n\to\infty} \mathbb{P}\Big[|Y_n^{(m)} - W^{(m)}| > \epsilon\Big] + \limsup_{n\to\infty} \mathbb{P}\Big[|W^{(m)} - W| > \epsilon\Big] = A + B + C.$$

Observe that from item 3 and 4 in the condition of Theorem 4, A and C will go to 0 when we let m goes to infinity. From item 2 in the condition of Theorem 4, B is identically 0. As a result, we conclude that $Y_n \xrightarrow{n\to\infty} W$. □

References

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